TOPICS IN LOGIC, INFORMATICS AND PHILOSOPHY OF SCIENCE

THE CHAIR OF LOGIC, INFORMATICS AND PHILOSOPHY OF SCIENCE
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CONTENTS

Witold Marciszewski

INTRODUCTION:
ON WHAT LOGIC, INFORMATICS AND PHILOSOPHY OF SCIENCE MAY HAVE IN COMMON

This volume presents the main interests of the team of researchers either working in or collaborating with the Department of Logic, Informatics and Philosophy of Science, University of Białystok, Poland. This Introduction hints at some issues of the volume as a whole, and against that background offers the abstracts of individual contributions.

Both the name of the Department and the title of this volume refer to some topics shared by the three disciplines, involved that is, logic, informatics and philosophy of science. They have a common core, and that amounts to a cluster of topics centered around the problems of formalization of reasoning, solvability of problems, decidability of theories. Each of these sciences approaches this fundamental problem in its own way. These issues are born in logic with Hilbert, Gödel, Turing, Post, Church, Tarski, etc. At the same time, they are concerned with algoritmic procedures in computer science, while their epistemological aspects are highly relevant to philosophy of science.

When taking into account the way in which the above scheme is being accomplished, the contributions to this volume can be divided into two classes. One of of them (items 1-6 reported below) deals with informatics, to wit its philosophical context (1, 2), its logical foundations (3), and some technical results (4, 5, 6). The remaining contributions take a formal approach to some philosophical and scientific issues; that approach is not directly related to informatics, but offers those prerequisities which are useful whenever computer applications are to be made.

* * *

1. The opening text by Witold Marciszewski consists of three parts. The first deals with the idea that modern science meets some postulates of the

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old Mathesis Universalis project. This is exemplified with philosophical problems in physics, and, in a more detailed manner, with mind-philosophical problems related to computer science. The latter issue is continued in the two remaining parts; the last suggests that as far as ontological assumptions of AI are concerned, one may take much advantage Leibniz's ideas.

2. It is Leibniz's approach with which Halina Święczkowska's discussion of the *idea of thinking as computation* is concerned. That approach is compared both with ideas of his contemporaries and with the modern AI projects.

3. The study by Edward Bryniarski, Andrzej K. Rogalski and Urszula Wybraniec-Skardowska deals with *the notion of truth in systems of knowledge* (related to the famous Tarski's achievements) in such an efficient way that the authors can sum it up with expressing the hope that the notion in question should provide foundations for algorithms to represent formalized systems of knowledge.

4. Anna Zalewska's report on *theorems proving in a certain sequent calculus* presents the author's result concerning a theorem-prover based on a Gentzen-like system of inference rules. Her result makes it possible to prove properties both of a sofware (in a relevant logic of programs) and of data structures (as dictionaries, natural numbers, etc). The author succesfully attacks some troublesome issues in the automated proofs research viz. the problem of finding a suitable schema of metainduction, and that of handling individual variables when operating with quantifiers.

5, 6. The technical report by Roman Matuszewski deals with *natural language presentation of formal mathematical texts*, meaning some texts formalized in a special formal language for automatic theorem-checking. Such texts are being then automatically rendered in ordinary English for the convenience of users of the system in question. The same subject is treated more thoronghly in the paper *A structural approach to human oriented presentation of formal proofs.*

7. The main objective of Anna Gomolińska's research in *rule complexes for representing social actors and interactions* is to build a general mathematical model of systems of interactive actors in a social environment; the construction of model starts from real systems as markets, families, political institutions, etc. Actors and their interactions are represented with complexes of rules whose tranformations provide us with a formal model of social processes (being an alternative to that originated with von Neumann's game-theoretical approach).

8, 9. Dariusz Surowik's two contributions report on his research in tense logic as based on two more fundamental systems, namely Jan Łukasiewicz's

multi-valued logic and that elaborated in Brouwer's intuitionism. The former demands rejecting the principle of bivalency, the latter – the tertium non datur law. The author's contention is to elucidate the indeterministic standpoint in philosophy with the use of formal means of tense logic combined in a way with intutionistic logic.

9. Andrzej Malec compares *two forms of legal definition*, to wit those expressed in the object language of a legal text and those expressed in the metalanguage of a legal text. The former are descriptions of legal constructions, the latter are stipulations on how to replace an expression in a legal text with another expression. This classification is compared with the traditional distinguishing between real and nominal definitions.

* * *

All the items in this volume, except for the first, are versions of papers addressed to various sections (including 'Logic and Computation' and 'Philosophy of Logic, Mathematics and Computer Science', both close to our Department's profile), and previously published as abstracts, of the 11th International Congress of Logic, Methodology and Philosophy of Science, August 20-26, Cracow, Poland. W. Marciszewski's Congress paper *Post's philosophy of logic meets cognitive science* is to appear in a volume to be published by Kluwer (Dordrecht); his contribution to this volume is a revised version of Introduction to the Internet journal *Mathesis Universalis*, no. 1 (www.calculemus.org).

Witold Marciszewski

Witold Marciszewski

THREE ESSAYS IN THE MODERN MATHESIS UNIVERSALIS

These sketches do not pretend to produce a complete picture of the modern revival of the 17th century programme of Mathesis Universalis; they are only to deal with some basic ideas*.

In the first of them, I am just to hint at a new relation between philosophy and science, which appeared in the 20th century. This relation resembles rather that postulated by the programme in question than that claimed by the followers of positivism and the so-called analytic philosophy.

The second essay addresses an issue raised by Leibniz, the greatest protagonist of Mathesis Universalis, to wit the question of whether logical algorithms can capture actual capabilities of human intelligence. The conjecture that they suffice to render any intelligent reasoning was characteristic of the Leibnizian project, while its modern version is more cautious, on account of the limitative results of Gödel, Church, Turing and Post. I do not discuss these well-known results; instead, I hint at two kinds of natural reasoning which seem to escape algorithmic procedures. This problem is continued in the third essay in which algorithmic procedures of reasoning are discussed under the title "mechanized deduction".

I. ON THE SCIENCE-ORIENTED PHILOSOPHY

Once upon a time people did not distinguish between science and philosophy. The Greek word *mathesis* (akin to the Sanscrit *manas* – the mind) denoted the whole of human knowledge, including what nowadays involves mathematics, science, and philosophy. Is there any chance of return to that lost state of intellectual happiness?

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The story of Copernicus, Galileo, Kepler, Newton, Leibniz, Einstein, Bohr, Heisenberg, Popper, Gödel, Tarski, etc suggest the answer in the affirmative. It can be summed up in the idea of *Mathesis Universalis* to denote the united forces of mathematics, logic, natural science, and philosophy.

Thereby one distances from those who decreed, like the Vienna Circle, an eternal divorce between science and philosophy. They were most angry with what they called "metaphysics". Up to now some people do believe that the only scientific attitude consists in the *Überwindung der Metaphysik*. A convincing riposte to that attitude is found in Karl Popper's phrase "metaphysical research programme", concerning empirical research.

1. The revival of bonds between science and philosophy

This revival can be exemplified by some ideas of two leading figures at the modern intellectual scene, to wit:

— the philosopher Karl Popper (1902-1994, *Logik der Forschung* 1934)

— the physicist Werner Heisenberg (1901-1976, Nobel award 1932).

Popper introduced the notion of *metaphysical research programme* in the context of views regarding relations between science and metaphysics. Scientific statements are apt to be tested in observation; metaphysical ones are not, but this does not imply a lack of meaning, as claimed by logical empiricism of the Vienna Circle, briliantly refuted by Popper (when tackling the question"who killed logical empiricism?", he used to answer "I fear that I must admit responsibility"). The core of Popper's anti-empiricism consists in his view that there is no theory-free observation, and no theory-free language, since facts have to be interpreted in the light of theoretical assumptions. Among them there are some most general ideas which are not apt to be tested by experience, but are necessary to guide it, as the idea that there is the world, that it evolves in a way, that it can be rationally investigated, etc. He called such ideas "metaphysical" and appreciated them as laying at the bottom of scientific research.

A nice example of relations between natural science and philosophy is found in Heisenberg's *Der Teil und das Ganze*, 1973. The book starts from saying that it is talks what gives rise to science – *Wissenschaft im Gespräch ensteht*, and then some talks with most brilliant physicists of our century are reported and commented. What is specially impressive in them, it is how these scientists were occupied with fundamental philosophical questions. Here are the titles of some chapters, to exemplify the point.

The concept of *understanding* in modern physics.
The quantum mechanics and a talk with Einstein

[theory vs. experiment, the principle of economy, simplicity of natural laws, etc.].
First talks on the relationship between natural science and religion.
Atom physics and pragmatist way of thinking.
Quantum mechanics and Kantian philosophy.
Discussions on the language.
Positivism, metaphysics, and religion.
Elementary particles and Platonian philosophy.

Let these examples hint at the intercourse between natural sciences and a certain type of philosophy, of which both sides take impulses for development. Let this type be called *Science-Oriented Philosophy* – SOP, for short. The revival of SOP in the context of the 20th century science is related to the scientific revolutions of our age. In the preceding century science seemed to be finally established with the mechanistic foundations as having no alternative. This quiet was disturbed by problems and discoveries exemplified by the above list of topics of Heisenberg's talks in which the typically metaphysical problem of the Whole (called *der grosse Zusammenhang* by him and his collegaues) must have emerged.

SOP is obviously related to *philosophy* **in** *science* with which it shares interest in philosophical foundations of sciences, and to that quest for scientific foundations of philosophy which may be aptly called *science* **in** *philosophy*. SOP has some topics in common even with *analytic philosophy* inasmuch as the latter attempts at clarifying philosophical concepts (and not denying their significance). Since some of them belong to the foundations of science, there results a common field of interest.

It is in order now to descend from these highly general considerations, and to focus upon an individual science germane to a selected part of philosophy. The science to be considered is informatics, while the relevant piece of philosophy provides a conceptual framework to deal with a relation between physical and abstract entities.

The term "informatics" is convenient on account of its conciseness as well as certain indefiniteness that makes it possible to adjust it to current discussion. Its meaning is sometimes identified with that of "computer science", sometimes it suggests a more extensive study of information, in particular information processing. The latter is authorized by the Penguin *Dictionary of Computers* (A. Chandor et al, London 1985) which offers the following definition:

> "*informatics* – the science or art of processing *data* to provide *information.*" Fortunately, the terms italicized (in the original) for their import express the key concepts of this discussion.

In the conceptual framework to be proposed both varieties of processing are so construed that they appear not only in machines made by humans but also in organisms and minds, hence the term "informatics" can comprise a number of disciplines. Among them there is logic (historically, the oldest) which deals with *truth-preserving processing of data and information.* Furthermore, there is computer science with its ramifications, including AI etc, and there is also genetics, neuroscience, etc.

A conceptual framework is a system of fundamental notions which organize one's thinking about the world; since such fundamental, or most general concepts are traditionally called philosophical categories, some authors prefer the term "categorial framework", but fundamentality is involved in the very notion of framework, hence the both terms are acceptable.

At the bottom of any metaphysical research programme there is a conceptual framework involving philosophical notions; it forms an essential part of what is fashionably called a paradigm for science – after Thomas Kuhn (1962). According to him, a paradigm is necessary to answer questions like the following. *What are the fundamental entities of which the universe is composed? How do these interact with each other and with the senses? What questions may legitimatly be asked about such entities and what techniques employed in seeking solutions?*[1]

2. A philosophical framework for informatics

Let the universe of discourse of informatics be called CU, ie the *Cybernetic Universe.* The entities being in CU are engaged in activities of *Information Processing*, IP, and *Data Processing*, DP. These terms denote two categories which are different though inseparable. Data are physical entities which are coordinated to information pieces being abstract entities; if the latter category is exemplified by numbers, then the former by digits produced from ink, chalk, magnetized spots, electric impulses, etc. When such a stuff is used to produce the inscription being a sentence, then the corresponding piece of information is what we call a proposition. Other

[1] Thomas S. Kuhn, *The Structure of Scientific Revolutions.* Univ. of Chicago Press, Chicago 1962, 1970. See p. 4ff. To exemplify the notion of fundamental entities, let it be noted that the transition from SOP of classical mechanics to that underlyuing quantum mechanics implies new categories of objects, e.g. what C. F. von Weizsaecker (in Heisenberg 1973, quoted above, Ch. on Kantianism, p. 146) calls an *observational situation referred to by a perception* ("Jede Wahrnehmung bezieht sich auf eine Beobachtungssituation"). Another example. Some biologists so behave as if they believed in the reasonable entity called Evolution, when speaking, e.g., of "a trick of evolution which has unlimited inventiveness. Even if this is just an informal way of speaking, it exemplifies a spontaneous creation of categorial framework; it replaces the older, prescientific, one (involving the concept of God), nevertheless it preserves the category of the Mind controlling Nature.

examples: a proof, an algorithm, a program, belong to the things subjected to IP while their physical records – to DP. Since IP and DP processes are inseparable, for the sake of conciseness the name of the former can stand for both, unless the problem discussed requires that they be distinguished.

Data and information are processed with informational *machines*; in the present context, the adjective should be regarded as obvious (hence negligible) because other kinds of machines, viz. tools (to process a material stuff) and engines (to process energy) do not belong to the domain in question (CU), even if sometimes their relation to informational machines should be considered.

Two categories of machines are studied in the sciences of information processing. Since no suitable modern terminology has been coined to distinguish them, let us employ Leibniz's terms: *natural machines*, namely organisms, and *artificial machines*, eg computers.

Natural machines, capable of perception and having purposes of their own (such as to survive, to reproduce, to form societies) are suitably equipped with devices for information processing corresponding to those purposes. Artificial machines are mere devices to serve purposes established by their producers and users.

A *machine code* (or machine language) is a coding system adopted in the design of a computer to represent the instruction repertoire of the computer. In a digital machine its expressions are sequences of symbols "1" and "0" recorded as states of a cell (an impulse, the lack of an impulse). This code is found at the lowest level of programming languages as mediating in communication between men and computers. The higher the level of a programming language, the closer this langauge is to users' language, and the more remote from the machine code involved; an intermediate language provides translations from a higher-level to a lower-level language.

Let us take this structure as a rough model of what is going on in a natural machine. Though we have just a partial evidence for that, such a working hypothesis should prove useful (even if finally refuted for the evidence gained later with applying it). Thus the concept of machine code will enter the categorial framework for Cybernetic Universe.

A partial evidence for the conjecture proposed is to the effect that there is a vast set of actions taken by an organism which succeed in solving a problem, or performing a task, without their subject being aware of them. Then the course of actions must have been controlled by instructions which do not belong to any language known to the subject in question (eg, if a driver moves the steering wheel without saying to himself "move the wheel", the instruction is supposed to have been written in an organic-machine code).

Provided that there exists a machine code in an animal, it should be taken into account in explaining intelligent behaviour. Let us consider this at the level of such a developed animals as are human beings. Intelligent action is one that leads to a success in problem-solving. Usually, humans solve a problem with a reasoning. To solve it, one has (i) to know premisses and (ii) to know how to process premisses in order to reach conclusion. If there is a problem, at least one of these two prerequisites remains unknown. Suppose, one does not know premisses (the same argument can be applied when (ii) is lacking). To find them, one must either address the cell in which they are recorded and stored, or to make a random search. Neither is being done in a conscious way, hence the search is steered by instructions recorded in a language which is not accessible to a conscious inspection.

Since the process of reasoning is a crucial factor in intelligent problem--solving, the research in intelligence should take advantage of theory of logic as dealing with validity of inferences. This is done in part II.

II. LOGIC AND INTELLIGENCE

There are three traits of an intelligent person, to wit those of being:

SYSTEMATIC – CRITICAL – INVENTIVE.

It is the first for which *astronomy* is listed in the curriculum designed by Plato for educating an elite: the contemplation of the perfect heavenly order should grant young people a similar order in their heads. The ancients used to associate order and wisdom, as witnessed by the maxim *sapientis est ordinare* (the wise's affair is to put things in order). Nowadays similar hopes are addressed to logic.

Logic, indeed, provides us with the highest ideal of *systemization* in the form of both deductive systems (ordered sets of statements) and some semantic models (ordered sets of things). Moreover, logic can resort to set theory, namely to the well-ordering principle which says that *every set can be well-ordered*; its validity in mathematics is widely acknowledged, while its validity in empirical domains can be fruitfully conjectured.

It is also logic to which we owe most precise standards of *criticism*, concerning arguments, definitions, classifications, methods of testing theories, etc. There is a lot of intelligent actions which prove so complex that these logical standards do not match that complexity, nevertheless logic forms a set of standards to be approximated as much as possible.

As for *inventiveness*, the contemporary logic, unlike the so-called logic of discovery as projected in various forms by Bacon, Descartes, Leibniz, etc., does not pretend to guide the process of discovering truths. On the contrary, among the greatest achievements of logic (due to the discussion on decidability of theories) there is the concept of *algorithmic* procedures. Since they are the opposite of creative behaviour, logic contributes to the notion of inventiveness by hinting at a necessary negative condition, namely, that for a behaviour to be inventive implies its being non-algorithmic. However, invention as well as other traits of intelligence should be assisted by algorithmic devices to increase its potential.

When an *algorithm* – an abstract mathematical object – is expressed in machine code, in which instructions for the machine are recorded, it becomes a *program* to control the work of the machine.

These concepts, fruitfully generalized, explain the behaviour of organisms too: one speaks of genetic programs, of instictive behaviour as being programmed, etc. Presumably the notion of machine code can yield a model to understand the functioning of central nervous system. Such a machine code in organisms deserves to be called an *internal language*, meant as one in which Nature records its algorithms to control animal behaviour.

If there is – as claimed by some authors – a Darvinian contest between ideas in an individual mind, a contest to develop this mind's intelligence, then an algorithmic equipment acts like environmental conditions to be met by the ideas in their quest for survival and development. To explore, though, that natural logical environment, we should go far beyond current logic.

1. Beyond the frontiers of official logic: How-Reasoning vs That-Reasoning Model-Based vs Text-Based Reasoning

There are reasonings which most logicians did not dream of. They used to tell us that logic copes with reasonings as truth-preserving transformations of sentences. There are, though, innumerable cases of inferences which (i) are *not* truth-preserving and (ii) do *not* depend on any texts.

Kind (i) is nicely exemplified by the so-called "problems" in Euclid which consist in making something out of another thing, say, *to describe an equilateral triangle on a given finite straight line* (Book 1, Problem 1). The property of truth-preserving cannot attach to the reasoning which solves such a problem because the solution does not consist in asserting a proposition; instead, it amounts to producing a *construction*. In other words, the conclusion does not says *that* there is so-and-so; it says, instead, *how* to construct a thing. Let such processes be termed as *that-resonings* and *how-reasonings*, respetively. Only the former were lucky enough to have become the subject-matter of official logic.

RESEARCH TASK 1. In which way how-reasonings and that-reasonings are related to each other? Sometimes, as seems to be the case in Euclid, it is possible to reduce the former to the latter, since the latter can be interpreted as ones yielding existential theorems (the success of construction proves existence of the object constructed). This supposition should be checked, and then one should find to what extent it can be generalized. Since methods of mechanizing that-resonings have become a matter of routine, indirectly this shuld pave the way to mechanizing how-reasonings.

Kind (ii) is most convincingly exemplified by reasonings carried out by animals as lacking any possibility of producing texts, while being capable of processing pictures. There is a widely known case of unverbalized problem-solving, namely that of Koehler's chimpanzee Sultan who fitted a bamboo stick into another, after many attempts to solve the problem of grasping fruit that was out of his reach. A human would react in a similar way, as it is the only correct solution, and he also would not need any verbalized inference. The whole inference can be done silently in one's imagination; it consists in processing mental images, or (more generally) *models* of some things, viz., the sticks and the fruit. Before the agent fits a bamboo stick into another, he tries this strategy in a wordless *Gedankenexperiment* to obtain the hypothesis that with an extended stick one would overcome the distance to the fruit. This encourages one to externalize this mentally modelled action in the form of overt behaviour. Let such processes be called *model-based* reasonings as opposed to *text-based* reasonings; only the latter have become the subject-matter of logical research[2].

RESEARCH TASK 2. As to kind (ii), ie the model-based reasonings, there is no doubt that many of them are not reducible to the text-based reasonings. Thus AI faces the task of *creating the method of recording models in an internal language and providing rules of processing them*, analogous to logical inference rules.

Still another step beyond the frontiers of official logic is needed in discussing how-reasonings and model-based resonings from the viewpoint of a theory of intelligence. It should consist in creating a theory of conceptual systems, or (more imaginatively) conceptual nets. The existing logic teaches us how to transform premisses into conclusions, but does not teach how to find premisses which we do not have yet. And this is the main problem in the both "unorthodox" kinds of reasonings. When constructing an object, one looks for a suitable stuff; and when dealing with models instead of texts, one has no ready verbalized premises; it is a conceptual net which should assist an efficient search for them.

[2] The terminology to distinguish this opposite kinds of reasonings is far from being established. The suggested pair of terms is found in *Mechanization of Reasoning* by W. Marciszewski and M. Murawski, 1995. The same opposition is rendered by the pair "*objectual inference – symbolic inference*" in the chapter "Reasoning, Logic, and Intelligence" of W. Marciszewski's *Logic from a Rhetorical Point of View*, 1994. The first pair is preferred here as the concept of (mental) model is very useful in the research in question.

2. Concepts as Meshes to Catch Premises

There is a strong evidence that the level of intelligence depends on one's set of concepts. The more such a set is like a system, that is, the more systematically arranged, the more it helps in finding pieces of information needed for reasonings in problem-solving. Clear examples can be found in sensory perception. If Sherlock Holmes perceives a lot of details relevant to the case under study, while Dr. Watson does not, and so Holmes proves more intelligent, this is because he has a reacher conceptual apparatus, necessary to put those questions which, in turn, guide his perception.

Obviously, there is a feedback between one's ability to invent and systematize new concepts by himself and a conceptual system acquired through learning: a more learned person is more inventive, and a more inventive one is more capable of learning new things. A similar feedback holds for the two remaining components of intelligence.

Not only the number of elements is what counts but even more their interrelations constituting a conceptual system. This is so because of the enormous definitional role of such relations – as shown in the methodology of deductive systems when it deals with implicit (ie postulational, axiomatic) definitions. Let us imagine, first, two separate conceptual systems A and B, and then a greater system

$$(A, B, [A*B])$$

where "[A*B]" stands for the set of relations of elements of A to elements of B, and vice versa.

Let A be a conceptual system concerning economics, and B – politics; let the concept "market economy" be in A, and "rule of law" in B. In the concept: "free Competition within the rule of Law", for short "C*L", "C" belongs to A while "L" to B. This combination modifies the content of each of them, eg "free competition" in a lawless society would mean something very different (rather the law of the jungle). One who has the concept "C*L" at his disposal is more than others capable of finding premises for reasoning about public affairs (eg in a debate on antitrust regulations).

Another example: one who is versed in computers, logic and neurology (as was, eg, John von Neumann) is better prepared to reason about each of these subjects than someone whose conceptual system is limited to just one of them. His advantage consists in being able to grasp more facts relevant to the problem in question, that is, in greater ability of finding premises.

One more example. An author proves intelligent *qua* author if he skillfully handles text partition (paragraphs, sections, etc.), indenting, formation of titles, hierarchizing of concepts (bold, italics), etc. Such intelligence may progress owing to a sophisticated software like TeX. Then new typographical means (hardly available with typewriter) as varieties of framing, itemization, etc. extend the conceptual system concerning logical structures of texts, and thereby structures of thought; this, in turn, makes one more versed in recognizing and creating such structures.

> RESEARCH TASK 3. Suppose a device should be constructed, e.g. an expert system, to support human intelligence in reasoning and deciding about a definite subject-matter. The task to be performed to grant the system high efficiency will include a *systematization of concepts which should approximate axiomatic systems* (as much as possible and necessary).

The old Leibniz's dream of *universal characteristic*, that is a system of concepts so arranged that mere combinations of signs guide problem-solving, to some extent may revive in a system forming the common core of all conceptual systems relevant to our civilization. It should involve fundamental notions of logic and set theory, arithmetic, physics, computer science, cognitive science, theory of human action, etc, properly systematized.

This may seem a crazy idea if taken without due provisos. However, if considered in the long run, and in view of enormous new possiblities of intellectual collaboration to be expected from Internet, and when taking into account the enormous potential of the method of Hypertextual links, this phantastic project becomes likely to approach reality.

III. MECHANIZED DEDUCTION SHOULD IT SERVE AS A MODEL FOR AI SUCCESS?

There is a long way from Leibniz's arithmetical machine to the mechanized reasoning carried out by provers and checkers. The final success is due to *physicalization of information* as well as *formalization and algebraization of logic.*

In a germinal form, physicalization started with mechanical contrivances like those of Pascal and Leibniz, but the real physical flesh it needed did not appear until the era of electronics. Formalization germinated in the medieval notion of logical form, to be by Leibniz combined with Thomas Hobbes' idea that reasoning is like computing. However, the logical tools to implement that idea have not been invented until in our century.

A next step toward AI should consist in mechanizing conceptualization. The achievements in proving crucial theorems, like those of Cantor or Gödel, are due to surprisingly new ideas. Were such concepts mechanically produced, we would enjoy an artificial Cantor, artificial Gödel, etc. This test is a bit more difficult than Turing's. Anyway, AI enthusiasts should meet the challenge, even if more demanding than that narrated by S. Lem in the poem "Electronic Bard" from *The Cyberiad* (translated from Polish, Mandarin, London 1990). Here is the story.

> Trurl and Klapaucius, supeinteligent robots, are constructors of intelligent machines, eagerly competing with each other. When Trurl boasted of his new AI masterpiece, Klapaucius suggested the following task to test its talents. "Let's have a love poem, lyrical, pastoral, and expressed in the language of pure mathematics. Tensor algebra mainly, with a little topology and higher calculus, if need be. But with feeling, you understand, and in the cybernetic spirit."
>
> The following poem, produced by the machine immediately after the constraints were put, exemplifies a sophisticated manner of concept formation, namely that resorting to *metaphors* (why not to use mathematical ones to express machine's erotic phantasies?). Only if the art of metaphor is mastered by a machine, it can be said to match humans.

Come, let us hasten to a higher plane,
Where dyads tread the fairy fields of Venn,
Their indices bedecked from one to n,
Commingled in an endless Markov chain!

Come, every frustum longs to be a cone,
And every vector dreams of matrices.
Hark to the gentle gradient of the breeze:
It whispers of a more ergodic zone.

In Riemann, Hilbert or in Banach space
Let superscripts and subscripts go their ways.
Our asymptotes no longer out of phase,
We shall encounter, counting, face to face.

I'll grant thee random access to my heart,
Thou'lt tell me all the constants of thy love;
And so we two shall all love's lemmas prove,
And in our bound partition never part.

For what did Cauchy know, or Christoffel,
Or Fourier, or any Boole or Euler,
Wielding thier compasses, their pens and rulers,
Of thy supernal sinusoidal spell?

Cancel me not—for what then shall remain?
Abscissas, some mantissas, modules, modes,
A root or two, a torus and a node:
The inverseof my verse, a null domain.

Ellipse of bliss, converge, O lips divine!
The product of our scalars is defined!
Cyberiad draws nigh, and the skew mind
Cuts capers like a happy haversine.

I see the eigenvalue in thine eye,
I hear the tender tensor in thy sigh.
Bernoulli would have been content to die,
Had he but known such *ax cos 2 phi*!

This essay is to discuss a source of advances of mechanized deduction (Section 1) and put the question regarding possibilities of mechanized conceptualization (Section 2).

1. The source of success: physicalization and logical resolution

To examine the relationship between information in general and the physical stuff in which it materializes, we need terms to denote some portions of either of them. Let them be coined according to the following terminological proposal (which does not repeat any existing conventions but does not oppose them, either).

> I shall speak of *data* as discrete physical records of information, for instance, symbols written down with pencil, configurations of electric impulses, of magnetized spots, etc. By the very fact of such recording, a continouous flow of information gets segmented, hence it is in order to speak of *pieces of information* assigned to respective data, for example the number zero assigned to the inscription "0". As shown in this example, information pieces belong to the domain of abstract objects.

In arithmetical machines, pieces of information are numbers represented by physical states, be it configurations of cogs, be electric impulses, etc. Hence

an indispensable way to physicalizing any thought consists in its arithmetization, that is, expressing it in numbers; for instance, in such conventional way as the numbering items of a vocabulary and then using numbers instead of words to express concepts.

The other part of physicalization consists in constructing logical gates, ie, physical devices (mechanical, electric, etc) to realize logical functions such as negation, conjunction, disjunction. The possibility of mechanizing reasonings is due to the idea of *logical function* which evolved from a historical process, going back to medieval attempts at establishing algorithms which would mechanically yield valid inferences. Arguments of logical functions are like atoms in the universe of proof procedures, while specially devised rules make it possible to resolve any proof into such elementary constituents. The so obtained set of atoms has some physical properties which automatically decide whether a sequence of strings is a valid proof[3].

The process of resolution has the algorithmic, or mechanical character owing to the fixed order of steps which is determined by the structure of formulas occurring in the proofs. Here arises the interesting *issue of the difference between humans and computers as far as the mode of reasoning is concerned.* Theoretically, people can reason in the same way, that is, using the resolution algorithm. But they never do, and have serious reasons to proceed in their own way. The situation may be compared to the difference between programming in a machine code and that in a high level language. People can do the former too, and sometimes, if necessary, actually do, but the efficiency of this method when applied by humans is so low that in a long run it would stop any progress in programming. Anologously, had mathematicians reasoned with the resolution algorithm, this would have stopped mathematical progress.

The usual way of human reasoning was discovered already by Aristotle and described by him as the search for a *middle term.* To endorse this view, we just need to understand this expression in a broader sense, as the search for intermediary links between assumptions and conclusion, if the entailment is not obvious. The linking proposition should be such that it be obvious both that if follows from the assumptions and that the conclusion follows from it. How many such links should be looked for, depends on what kinds of following are regarded as obvious. In this respect there are

[3] A conspicuous description of resolution procedure addressing beginners is found in Patrick Henry Winston's *Artificial Intelligence*, 2nd ed. Addison-Wesley, Reading (Mass.) etc 1984. The same subject in historical arrangement is discussed by W.Marciszewski and R.Murawski in their *Mechanization of Reasoning in a Historical Perspective"*. Editions Rodopi, Amsterdam 1995, in Sec. 7.3.

notable differences between individuals, groups and cultures, as indicated by historians of mathematics[4].

Interestingly, such subjective differences can be simulated by automatic provers or checkers with varying degrees of obviousness. For instance, one checker accepts (ie "regards as obvious") the joint use of ponendo ponens and instantiation in a single step, while another one, instead, requires two separate steps.

Fortunately, in the first-order predicate calculus such differences in logical intuition do not harm the objective validity of logical rules since there holds equivalence between the method of resolution, as used by computers, and the method using classical inference rules, suitable for humans, as ponendo ponens, substitution, etc. This is a great success in that field of AI research that we have methods of mechanized proving and proof-checking and, moreover, enjoy the certainty that human and mechanical reasoners can assist each other in reasoning. Is it possible to repeat such a success in other cognitive activities, particularly in concept formation, or conceptualization?

2. A project of machine-oriented research in conceptualization

Some rudiments of conceptualization are not unfamiliar to robots, for example, they can learn class descriptions from samples. However, if machines are to assist humans in a way comparable to mechanized reasoning, much more is required. In particular, they have to see such connections as those appearing in the use of pictures and metaphors at the very start of creative scientific thinking. Not less important is to perceive connections between various objects of the same domain.

To explain the latter, let us take advantage of some basic notions of Peano's arithmetic: "number", "zero", "sucessor", and "addition". In order to state, or even to understand, axiom 1 (that 0 is a number) one has to see a relation between 0 and the set of numbers; after he realized it, he

[4] Cf. Raymond L. Wilder, *Mathematics as a Cultural System*. Pergamon Press, Oxford 1981. The following passage of this book (p. 40) is concerned with historically changing standards of exactness. "It is ordinarily asserted that mathematical proof rests on *logic*. Ideally, this may be correct; actually it is generally incorrect. Analyze the general run of mathematical proof and it will be found to contain hidden assumptions [...] generally accepted in the contemporary mathematical culture. The classic example is that of Euclid, whose geometry was for centuries held up as the ideal example of rigorous proof. We know now that it contains geometric assumptions, unstated, that invalidate some of the proofs." This is a thought-provoking comment though it goes too far in its relativism; there were, indeed, some mistakes in Euclid, which were not detected until logic provided most reliable means for checking proofs, but owing to this logic such failures should not occur in the future.

understands both what 0 is, and what a number is, better than he did before realizing the connection. To proceed further in constructing the theory of arithmetic one must perceive the connection between the function of successor and that of addition (the former being a special case of the latter). Again, the understanding of the nexus increases the understanding of the nature of its members, though at some earlier level the latter was a prerequisite for the former (as is also the case in the previous example).

Thus we encounter a remarkable and very important cognitive feedback between understanding a whole and understanding its parts. This and some previous observations hint at what should be fed into a machine to make it capable of producing and using concepts, namely.

(1) A set of *logical and ontological categories* which are like tools for producing concepts of any kind whatever. Such are categories of class, membership, equality, equivalence relation, ordering relation; thing, property, fact, event, state, whole, part, etc.

(2) Moreover, the machine should be equipped with a suitable *knowledge* concerning the domain in question, eg arithmetic, medicine, economics.

Items from the first list are those which are involved in defining connexions crucial for forming, developing, and understanding concepts. This is why such intellectual equippment is an important component of intelligence.

However, before one properly defines a connection, one has to notice it. Such *intellectual perceptiveness* is an enormously important factor of intelligent conceptualization; it belongs to what in another place is called inventiveness. Its results should be critically assessed, and then imbedded into a system, but it is up to it to yield what is necessary to start with.

This is a substantial hint for an engineer who would build a conceptualizing machine. Besides a vast memory, it should possess an enormous ability of scanning all the relevant items stored in the memory (like Trurl's electronic bard who must have searched two enormous repertoires of words to match their erotic and mathematical meanings and, morever, to match sounds for rhyming).

It can be easily noticed that the ability of rapid subconscious scanning is not evenly distributed among human minds. There must be a mechanism responsible for these variations; its examining should be involved in projects of constructing artificial minds. In fact, at least part of the problem reduces to such physical parameters as speed of signals and density of network connections. Optimalizing of these two factors should form a major task in cognitive engineering.

Productivity is a necessary but by no means sufficient condition of intelligence. It is not enough to produce a lot of inferences or a lot of concepts.

The products should be somehow "good". Owing to formalization, we have methods of critical and precise assessing of deductive reasoning; it reduces to the mechanically testable validity, eg by applying resolution. Unfortunately, no such direct answer is available with regard to conceptualizations.

However, we can try an indirect way, even if less reliable, which leads to the goal via mechanization of reasoning. Again, the example of Peano arithmetic is to render services. We have grounds to believe that the concepts occurring in its axioms are cognitively good because we obtain an enormous set of consequences, and we did not find any inconsistency among them. Moreover, there exist consequences of a colossal cognitive value. It is the set of axioms which should be praised for these merits, and the same appraisal is due to the concepts involved. Thus we obtain the paradigm of an indirect checking of cognitive value of concepts. Such a paradigm is really functioning in physics and other empirical sciences, with the additional feature that among the conclusions deduced there are observational statements.

At the very last, it is in order to put a metaphysical question, that concerning possibility of artificial intelligence which would match, or even surpass, natural intelligence in reasoning and conceptualization. The answer is not without an impact on practical research, since it may influence the statement of goals, and – obviously – it is interesting by itself. Among varying answers there is one relatively little known to philosophers engaged in the AI questions, namely that suggested by Leibniz, in that stream of his thought which is represented by *Monadology.*

Metaphors and pictures appear as sources of scientific concepts at first, most inventive, stages of creating a scientific theory. How, for instance, Niels Bohr was aware of that with regard to his atom model, is reported in a remembering by Werner Heisenberg. Bohr is said there to have said the following.

> We should clearly see that the language can be used here just like in a literary narrative which does not pretend to describe facts in an exact manner; it tries, instead, to produce pictures in the minds of audience and thereby to establish some connections between thoughts. (Werner Heisenberg, *Der Teil und das Ganze. Gespräche im Umkreis der Atomphysik.* DTV, München 1993. Ch. 3, p. 14; ad hoc translation by this author.)

If Bohr is right, then AI is to meet an enormous challenge. It is worth trying since its failure as well as success should highly increase our knowledge of mind.

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Halina Święczkowska

LEIBNIZ'S IDEA OF THINKING AS COMPUTATION

The idea of automation of reasoning in science, which is extremely high-ranking nowadays in connection with the development of computer technology and research into artificial intelligence has been present in the history of mind since taking up a cohered with it idea of mechanization of reasoning. It is therefore proper to mention here the greatest enthusiast of this idea on the threshold of the contemporary science, who treated the idea of mechanization in reasoning very widely, assuming that every mind process can be expressed in some physical manner – Gottfried Wilhelm Leibniz.

Leibniz presented the concept of the universe as a harmonious system, where both unity and diversity, coordination and division of elements can be found; and this wonderful order can be delivered from the fact, that nature is God's Clock (horologium Dei)[1].

Although his model of nature was based on the laws of mechanics but, by accepting this begining Leibniz emphasised that those laws do not depend on the mathematical extension but on some metaphysical causes[2]. In his opinion, the basic scarcity in mechanistic physics did not take into consideration some dynamic elements existing in nature[3]. By replacing the Cartesian principle of maintaining motion by the principle of perfect harmony between the cause and its result, Leibniz educed metaphysical consequences. Namely, that power or energy, however measurable by its future effect, is something real, constantly existing in substances[4].

[1] G. W. Leibniz, "Die philosophischen Schriften von G.W. Leibniz", 7 vol., ed. G. E. Gerhardt, Berlin, 1875-1890, vol. I, p. 25. (Later as GP, volume and page).

[2] GP II, 58.

[3] G. W. Leibniz, "Leibnizens Mathematische Schriften", 7 vol., ed. C. I. Gerhardt, Halle, 1849-1863, vol. VI, pp. 117-123. (Later as GM, vol., page).

[4] GM VI, 233-254. See also P. Costabel, "Leibniz et la dynamique", Herman, Paris 1960, pp. 97-106.

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In the language of Leibniz's metaphysics the conception of power means endeavour or conatus – one of the basic attributes consisting of an elementary unit of a being, called by Leibniz as a substance or a real atom of nature[5].

The world of nature is built from indivisible metaphysical points – substances are arranged according to the principle of certain calculus:

"WHEN GOD CALCULATES AND EXERCISES HIS THOUGHTS THE WORLD IS MADE" (*Cum Deus calculat et cogitationem excercet fit mundus*)[6]. It has phenomenal status but it is the world of 'well established phenomena'[7].

From this point of view, every single look at this world, every perception is in a perfect harmony with the perspective of God, and the genuineness of 'God's vision', which constitutes warrant of genuineness of perception of substances, created by God[8].

Only human mind has the ability of conscious inquiry into the universal order of being equiped in the natural order of ideas, isomorphous with God's scheme of creation. For Leibniz, the human mind is similar in its nature to God's, varrying only in perfection of action[9].

The natural order of ideas, according to Leibniz, founds common linkages between angels, humans, and all intelligent creatures which establishes the natural order of cognition that should be followed, withstanding, the essential limitations of the human species[10].

The limitations that all minds created by God are subordinated to cause by experiencing direct contact with the world of ideas, in only few cases. Leibniz indicates here only mathematical cognition[11]. The language sign which is essential for maitaining and supporting perceptional functions, records the actions of the mind on the conscious level, which allows to catch and record the results of these actions in some physical form. The language is

[5] G. W. Leibniz, "Monadology", § 3, in "G. W. Leibniz – Philosophical Papers and Letters", ed. L. E. Loemker, 2nd ed. Dodrecht: Reidel, 1969.

[6] G. W. Leibniz, "Dialogue", in "Gottfried Wilhelm Leibniz – Philosophical Papers and Letters, ed L. E. Loemker, p. 185.

[7] GP II, 304, GP VII, 322.

[8] G. W. Leibniz, "Discourse on Metaphisics", in G. W. Leibniz, "Philosophical Papers and Letters, ed. L. E. Loemker, pp. 311-312.

[9] G. W. Leibniz, "Monadology", § 83.

[10] G. W. Leibniz, "New Essays on Human Understanding" III, 1, § 5. ed. P. Remnant and J. Bennett. Cambridge: Cambridge Uniwersity Press, 1981.

[11] GP IV, 423.

secondary to the actions of the mind, but the perception, according to Leibniz, is impossible without it[12].

Leibniz assumed that aperception is reasoning that leads to an idea. This aperception has an algorithmic character, which he named arithmetical, because Leibniz did not use the term 'algorithm'. The natural order of ideas is isomorfic with the order of universe and this is a consequence of the Creator's actions, who has chosen the best of all possible worlds, creating it according to the inner order of his thoughts. God's order is a mathematical order – it reminds us that when 'God counts the world comes into being'. The harmonious system of the universe can be described only by a language which reconstructs the natural order of ideas. Discovering or building such a language nurtures the development of knowledge, so as in pure mathematics, there would be no place for competitive theories. Inspired by mathematical symbolic representation and the effectiveness of algorithmic procedures in mathematics and syllogistics, Leibniz was fully convinced that the possibility of creating a perfect language really exists[13]. Analogous to God's order, language allows the ability of the mind to overcome substantial tendency of making mistakes.

Although Leibniz had many predecessors also searching for the perfect language system, there is still something that makes his vision of universal language unique. Thus, Descartes indicated the main direction of research, but by questioning the cognitive usefulness of a language in general, he set aside the need of creating a universal language. Understanding a sign in a purely conventional way, he proclaimed that the material side of the sign has no corelation with the meaning of it, what stood in opposition to the usefulness of expressions in reasoning. It ment that reasoning as an operation on ideas cannot be supported by anything material[14]. Cartesian methodology reffered to the direct contact of the mind with an object of cognition. Leibniz's methodology indicated to the indispensability of another way of thinking in which the contact of the mind with the object of cognition is not direct. This process takes place by using the intermediary of signs as those instruments of reasoning, which represent the object assigned to them. The

[12] G. W. Leibniz, "Letter to Walter von Tschirnhaus", in. G. W. Leibniz, "Philosophical Papers and Letters, pp. 192-195. See also M. Dascal, "Leibniz: Langue, Signs and Thought", John Benjamins Publishing Company, Amsterdam/Philadelphia 1987, pp. 42-43.

[13] GP VII, 3-42. See W. Marciszewski, R. Murawski, "Mechanisation of Reasoning in a Historical Perspective", Rodopi, Amsterdam-Atlanta 1995, pp. 103-114.

[14] See J. Kopania, "Funkcje poznawcze Descartesa teorii idei", Białystok 1988, pp. 222--230.

signs should comply with one requirement, according to Leibniz's theory of representation, they should express that object[15].

The indirect way of thinking about things, where an idea of a thing is not revealed as an object of aperception, was called by Leibniz, a symbolic or blind thinking[16]. The conviction is that in its essence every action of the mind consists of counting. Athough this idea was borrowed from Hobbes[17], Leibniz gave it a completely new dimention. He assumed that thinking which leads to aperception is analogical with some kind of counting. Accordingly, by substituting actions of mind on every of its stage by symbolic representation will help to reconstruct the whole perception process and will formulate a proper system of rules, which govern thinking.

The foundation of algorithmic procedures lies in a purely formally defined language, which refers only to the physical shape of expressions and their combinations. Undoubtely, such a language would be ideal, as Leibniz wished it to be. He wrote:

"The progress of the art of mind's invention considerably depends on the art of composing signs (...). If there was created either a precise language (called by some the Adam's language) or at least some kind of truly philosophical alphabet, with help of which the notions would be reduced to human thought's alphabet, everything that can be reached by mind on the basis of data could be obtained by specific counting in the same way as problems of arytmetics or geometry are solved."[18]

The system acquired in such a way would become "a sensorial and somehow mechanical guide of mind, comprehensible even for the least clever. As following the text, thinking will proceed gradually, a written text will be a drift for thought."[19]

What's more, it would allow to settle the truth in a purely mechanical way, "the truth would be visible as in the picture, as if printed by a machine. It would happen due to the fact that criterion creating the truth would be arranged in a mechanical way, making it visible."[20]

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15 See H. Święczkowska, "Language as the Mirror of the Mind", in "On Leibniz's Philosophical Legacy in the 350th Anniversary of His Birth", Series Studies in Logic, Grammar and Rhetoric 1(14), Białystok 1997, pp. 13-34.

16 GP IV, 423.

17 GP IV, 64.

18 GP VII, 198-199.

19 GP VII, 14.

20 GP VII, 10.

The historians of science indicate that Leibniz's project of a universal language, not only did anticipate the Hilbert's programme of formalisation of mathematics but some modern research on artificial intelligence as well[21]. At the same time the language postulated by Leibniz, as reproducing mechanism of human perception was not limited only to mathematics, but is used in all domains of science. This would allow to solve all problems by using calculus and would later lead along the path of truth.

There are still some essential difficulties in reconciling Leibniz's trust confidence in miracolous force of algorithm with the dynamic structure of cognition unfolded in "Monadology". He wrote: "It must be confessed moreover that perception and all that depends on it are inexplicable by mechanical reasons that is figures and motions."[22]

Mechanization of cognition process is therefore only imitating nature, it is the human ability of reconstructing God's order, likewise as "no machine made by human art is not a machine corresponding with the God's one which is a natural authomaton."[23]

Approximate manner of reasoning in the matter of deciphering "the nature's code" can be found at von Neumann's. He wrote that language is considerably a matter of historical chance. Basic human languages are conveyed by tradition in various forms, but multiplicity of these forms in itself shows that they do not contain anything absolute or indispensable. Since ethnic languages are historical facts and not absolutely logical necessities it is quite reasonable to assume by analogy, that logic and mathematics are historical, random forms of expression that can exist in other forms that we are used to. For example, logic and mathematics of the central nervous system – if we investigate them as languages – they will appear to be structurally different in their type of languages known from everyday experience. In consequence, external form of our mathematics do not have absolute importance, from the point of view of inquiry what the mathematical or logical language used by the central nervous system really is[24].

Searching for the method rendering of God's order, Leibniz used, as he asserted, tools given by circumstances and events in historical process of cognition. These tools were – ethnic languages, mathematics and logic.

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21 See W. Marciszewski, "Why Leibniz should not have believed in 'filum cogitationis" and "From the mechanization of reasoning to a study of human intelligence" in "Studies in Logic, Grammar and Rhetoric", XII/XIII, Białystok 1993/94, pp. 5-60.

22 G. W. Leibniz, "Monadology", § 17.

23 "Monadology", § 64.

24 J. von Neumann, "The Computer and the Brain", New Haven 1959.

Granting the right to imitate creativelly and to cognite for a man, he belived that a man is capable of decoding the language of the nature due to a proper system of signs, providing that the isomorphism of things and signs was mantained. For as he wrote:

"...although characters are arbitrary, their use and connection have something which is not arbitrary, namely a definite analogy between characters and things, and the relations which different characters expressing the same thing have to each other. This analogy or relation is the basis of truth."[25]

[25] G. W. Leibniz, "Dialogue", in G. W. Leibniz, "Philosophical Papers and Letters", ed. L. E. Loemker, p. 184.

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THE NOTION OF TRUTH IN SYSTEMS OF KNOWLEDGE

Abstract

This paper corresponds to earlier research on truth and denotation of Wybraniec-Skardowska (1989, 1991, 1998, 1999) and Rogalski (1999). The authors present a formal theory of truth. It is not only an extension of Tarski's theory of truth, but differs from this theory in that to a large degree it can be employed to formulate a definition of truth for any system of knowledge, not only formalised ones. Knowledge in such systems can be represented not only in formal but also natural language. The theory presented is sufficiently general that various definitions of truth can be formulated within it, including syntactic and semantic definitions. The authors undertake an interpretation of the equivalence of these last definitions. They concentrate on the notion of logical truth and formulate several theorems about the relation of truth to language.

1. Introduction

So much has already been written on the notion of truth, and so many regulating definitions of this concept given, that one might ask whether we need one more definition, which would be a logical explication of this notion. Reflecting on the knowledge pertaining to truth and different approaches to truth (cf. Aristotle; Frege 1918/19; Tarski 1933; Carnap 1942; Davidson 1967, 1968, 1984; Hintikka 1991; Woleński 1990, 1993, 1999; Woleński and Simons 1989; Jadacki 1996); Soames 1984, 1999; Peregrin 1999) of the past two millennia, the authors have noted that de facto there still exists *no formal theory of truth* in which the legitimacy of various definitions of truth

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(most often the truthfulness of sentences) could be verified. In this paper we attempt to outline the formal-logical basis of such a theory. This theory is built upon Zermelo-Fraenkel set theory denoted by ZF.

In the context of the formal definition of truth, this papers takes into consideration only two such definitions: the syntactic and the semantic, in Tarski's sense (1933). The definition given by the authors explicates the notion of truth, a notion which can be defined very imprecisely using the following metaphors: "transparency", the "non-secretiveness" of things, "accessibility to the mind", or "the light making things visible to the mind".

Usually the definition of truth is attributed to Aristotle. Very well-known are the words of Aristotle (in his *Metaphysics*) cited by Tarski (1933): "To say of what is that it is not, or of what is not that it is, is false, while to say of what is that it is, or of what is not that it is not, is true." (see Aristotle *Works*, vol. 8, 1908). Aristotle also wrote: "This [truth and falsity] depends, on the side of the objects, on their being combined or separated, so that he who thinks the separated to be separated and the combined to be combined has the truth, while he whose thought is in a state contrary to that of the objects is in error." (see Aristotle *Metaphysics*, θ (IX), 10, 1051b, 1908). Saint Thomas Aquinas gave this formulation the form of a definition formula, citing the following passage from the book of Isaac Ben Salomon *De definitionibus: Veritas est adaequatio intellectus et rei, secundum quod intellectus dicit esse quod est vel non esse quod non est (G.G.I. 159; De veritate 1,2)* – loosely translated – "Truth is the agreement between the intellect and things, when the intellect says that something is and that thing really is, or that it isn't and in fact it isn't."

At the end of the 19th and beginning of the 20th centuries, this formula was many times criticised for its lack of clarity, although it was relatively easy to make it more precise. This was done, for example, by K. Ajdukiewicz (1927, 1949, p. 31), who formulated the definition of truth as follows: "The thought m is true means that thought m states that such and such is, and in fact, such and such is." (authors' translation). Roman Ingarden (1957, p. 374) introduced an interesting explication of this definition in application to propositions. According to him, a proposition is true if the state of things determined by the content of this proposition holds independently of the existence of this proposition within that domain of being in which this proposition locates it. Tarski (1933) gave the most concise form of this formula corresponding to the syntactic definition of truth. Tarski did not, however, accept this expression as a definition, but only as a definition postulate (cf. Tarski's well-known convention (T)):

Sentence E is true if and only if e,

where "sentence E" denotes the name of any concrete sentence, and "e" denotes a concrete sentence which is the designatum (reference) of this name.

Tarski presented the semantic definition of the concept of truth for formalized language. The method employed by Tarski proved to be very useful in metalogical research. Many difficulties were encountered in the application of this method in constructing a definition of the truth of a sentence of natural language when this sentence expresses knowledge about reality. Although many years have passed, it seems to the authors that there has not yet appeared a satisfactory definition of truth, which could be useful for systems of knowledge. And it is known that knowledge can be transmitted not only by means of expressions of natural language but also using other media, e.g. images, animated films, computer recording (computer coded).

2. A conceptualisation of knowledge about truth

For thousands of years scholars, not only from the circle of European culture, have thought that in their scientific research they have been "searching for truth", that their scientific dissertations have "contained truths", or that the scientific research results expressed in of a given science are "true sentences". Epistemological reflection on the variety of meanings of the terms 'truth' and 'truthfulness' appearing in scientific and philosophical literature leads to the following conceptualisation of knowledge about truth:

- **Truth** or **a system of truth** is that which for man makes things "transparent", "non-secretive", and "visible" to the mind.
- Man does not create truth, but takes part in the search for truth, that is, takes part in truth or remains outside of it.
- Truth exists independently of human will, but whether man takes part in the search for truth or not depends on the will to search for it. Metaphorically: man finding himself on the path to truth can take part in truth.
- That which makes things accessible to man is *a priori* included in his biological, philogenetical and cultural development, which directs him toward truth.

Henceforth we will limit ourselves in the description of truth only to rational aspects of cognition, thus we will aim to answer the question: When are the results of cognition in accordance with objective reality? We will omit also the problem of truth pertaining to things possessing the same features, i.e. repeating (the cognition of **extensional structure**) or represented by the same knowledge (the cognition of **intensional structure**). This means that we will refer truth to one cognitive process (the **act of cognition**). Therefore we assume that:

- Taking part in truth encompasses three domains of being: **objective reality**, a **system of knowledge**, and **structure of objective reality** (see Diagram 1).
- **Structure of objective reality** establishes all **states of things** that there are in objective reality.
- **States of things** are all collectively co-occuring things forming ordered connections, with or without man's participation. In a formal approach, states of things can be treated as distinguished things, or n-tuples of things ($n > 1$).

Take part in the truth

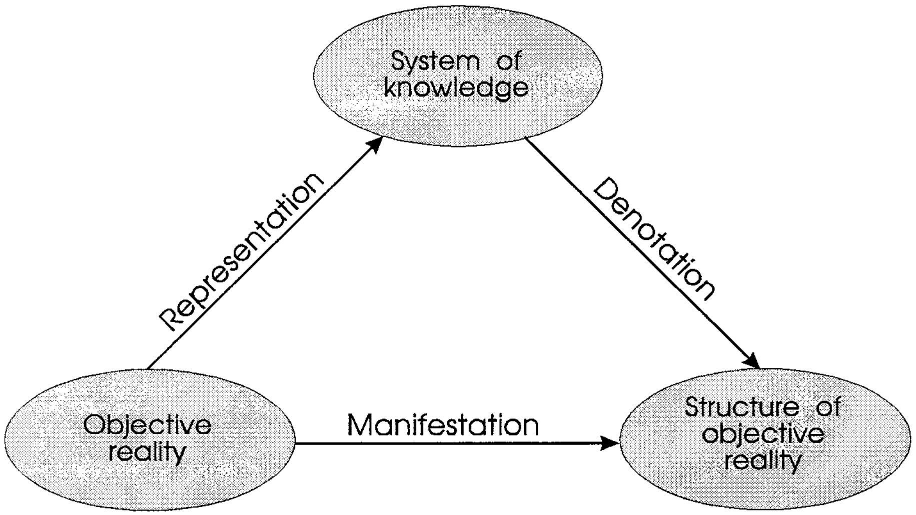


Diagram 1

- **A system of knowledge** is defined by means of three domains of man's cognitive activity: **cognitive reality, base of knowledge**, and **cognitive structure**.
- **Cognitive reality** is reality formed by man in culture-creating and cognitive processes, i.e. "humanised objective reality".
- **Knowledge** is a mirroring of states of things that are in man's cognitive reality in man's cultural and psychophysical mechanisms. It is not, then, identical to carriers of information in media, and can only be expressed in different sign systems, including natural or artificial (e.g. computer) language. The same knowledge can be expressed in different ways.
- **Knowledge** referring to things is defined by **terms**, and knowledge, which forms or formulates knowledge about things, is defined by **formulas**. In a system of knowledge expressed by a formalized language terms are terms of the language and formulas are its formulas. Elements of knowledge, i.e. **units of knowledge** create a **domain of knowledge**.
- **Findings are** formulas satisfied in cognitive reality by ascribing to them some state of things.
- **A base of knowledge** is a system forming knowledge by means of **operations** and **rules** on the basis of terms and formulas (**findings**) representing cognitive reality. These operations define new terms or formulas on the basis of other terms or formulas, as rules **derive** from one unit of knowledge other such units.
- A base of knowledge can be extended to **a metabase of knowledge**, to which belongs knowledge called metaknowledge, i.e. knowledge that a given knowledge is knowledge about states of things that are in cognitive reality, in other words, that **knowledge is satisfied in cognitive reality by ascribing to it suitable states of things** (i.e. **by valuation** and **evaluation**; see Diagram 2).

Take part in the system of knowledge

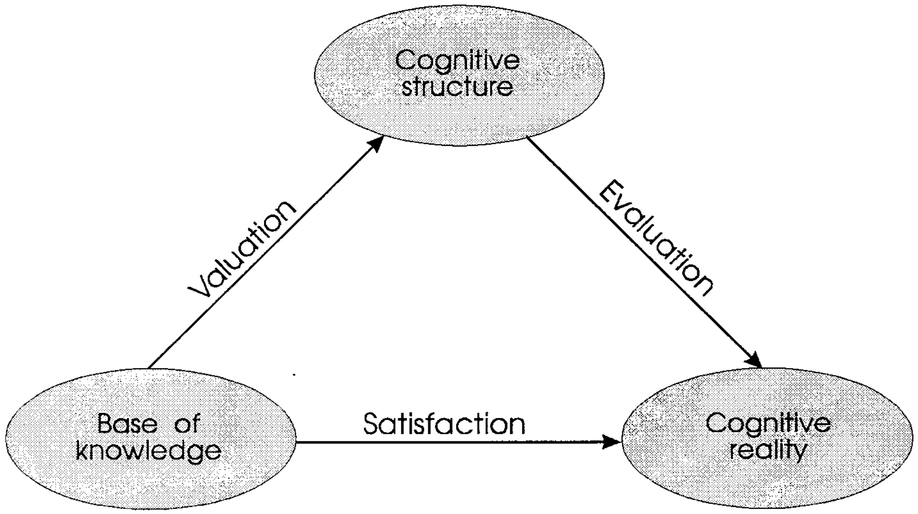


Diagram 2

- **Cognitive structure** establishes all states of things defined by metaknowledge.
- **Situational context** is a partial function given *a priori*, which uniquely binds terms, formulas, rules and functors, defining knowledge in a **subsystem of a base of knowledge**, with the counterparts of these components of this subsystem: **things, features, relations** between things, and **operations** on things, located in subsystems of objective reality that are accessible to cognition (which is not necessarily a **subsystem of cognitive reality**).
- The set of all situational contexts establishes the **representation**, which assigns to subsystems of objective reality that are accessible to cognition a **reflection of objective reality in the system of knowledge** (i.e. a subsystem of the base of knowledge). Only units of knowledge representing objective reality are **findings** satisfied for a given situational context in a subsystem of objective reality that is accessible to cognition.
- **A reflection of objective reality** is the entirety of terms and formulas derivable in a base of knowledge on the basis of findings and terms representing objective reality in the same situational context, i.e. it is a subsystem of a base of knowledge establishing the entirety of terms and formulas satisfied for a given situational context in subsystems of objective reality that are accessible to cognition, so called **cognitable reality**.
- The situational context limited to terms is **the reference** of these terms to concrete things.
- **Denotation of knowledge** is a partial function, given *a priori*, which for a given situational context allows us to assign a state of things belonging to the structure of objective reality to terms and formulas belonging to a given reflection of reality. Denotation of knowledge assigns to units of knowledge the set of situational contexts in which these units of knowledge are satisfied in objective reality, and assigns to a unit of knowledge appearing in these situational contexts a state of things about which the given unit of knowledge informs.
- **The manifestation of objective reality** establishes for a given context the state of things, which are the result of the denotation of some knowledge.
- **Units of knowledge** take **part in the truth** (see Diagram 1), if for a certain system of truth they are findings in the system of knowledge or for a certain situational context they satisfy one of the following conditions: they represent objective reality, they possess a denotation, they inform how objective reality manifests in this context.

- **Unit of knowledge take part in the system of knowledge** (see Diagram 2), if in the process of **valuation** states of things belonging to the cognitive structure are ascribed to them in such a way that is the process of evaluation, that is in the process of checking the agreement of this assignation with cognitive reality, the assignation was in accordance with the condition of **satisfaction**. Satisfaction for a given valuation is in this sense the composition of this valuation and evaluation.
- **Units of knowledge are true** if they take part in a system of truth. Then, they are satisfied in cognitive reality for a system of knowledge or they are satisfied in a subsystem of objective reality that is accessible to cognition (a **cognitable reality**) or for a situational context they have the denotation.
- **Truth is semantic** if formulas belong to its domain of knowledge.
- **Truth is syntactic** if formulas do not belong to its domain of knowledge.
- **Units of knowledge are semantically true** if they take part in a semantic truth.
- **Units of knowledge are syntactically true** if they take part in a syntactic truth.

3. Theory of truth

Let us proceed now to a formalisation of the above-given conceptualisation of knowledge about truth. First let us postulate that the sole primitive notion is the notion of the **universe** U, consisting of all **things**[1]. We assume that the universe is a nonempty set.

We assume that the formal representations of things in the mind are individuals, sets of individuals, families of such sets, etc. So it is natural to build a formal theory of truth upon on Zermelo-Fraenkel set theory. We also accept the set-theoretical formalism for the below-given presentation of the theory of truth.

Presenting the basics of the formal theory of truth we extend the ZF set theory adding to it terms and formulations introduced in the process of conceptualisation of knowledge about truth. They will be defined by means of definitional abbreviations of the formulas of the language of ZF theory.

[1] We understand **thing** very broadly here. A thing is that which has arisen, which is distinct and relatively stable in the worldwide processes of creation, which was in the past or will be in the future, which consists of diversity, changability, cooperation, or repetition of beings. Simultaneously, a thing is that which we want to know when we get to know the truth, and which usually we can give a name to. The world defines the existence of things, establishing unambiguously the set of all things: the universe U.

Terms will always denote some sets or elements of these sets. Generally, we assume that in the presented theory, every set is a subset of the universe or it is constructed from some subsets of the universe. Because of the complexity of definitional formulas, some definitions will be given in descriptive forms.

3.1. Reality

What things are, i.e. what **features** they have, what relations they enter into with other things, what they are **transformed** into in accordance with **operations**, or how they are distinguished as **individuals** depends on what **reality** they participate in. For example, some things of a cultural reality are tools while in a reality described by physics they are bodies. According to intuition, a reality creates a whole. So, by reality, in which features, relations, operations, individuals define things, we understand such a reality that any things in it possess certain features or are in some relations, or are results of the application of some operations, or are some individuals. Otherwise, if features, relations, operations or individuals do not define things, the reality is limited only to the set of all things belonging to this reality.

Definition 1.a

Let $\boldsymbol{Re} = \langle \boldsymbol{U}^{Re}, \boldsymbol{F}^{Re}, \boldsymbol{R}^{Re}, \boldsymbol{O}^{Re}, \boldsymbol{I}^{Re} \rangle$ *be an ordered system, where* $\boldsymbol{U}^{Re}$ *is a non-empty set,* $\boldsymbol{F}^{Re}$ *– the family of distinguished nonempty subsets of the set* $\boldsymbol{U}^{R}$*,* $\boldsymbol{R}^{Re}$ *– the set of relations on* $\boldsymbol{U}^{Re}$*,* $\boldsymbol{O}^{Re}$ *– the set of functions defined on* $\boldsymbol{U}^{Re}$*,* $\boldsymbol{I}^{Re}$ *– a distinguished subset of the set* $\boldsymbol{U}^{R}$*.*

a. *if* $\boldsymbol{F}^{Re} \cup \boldsymbol{R}^{Re} \cup \boldsymbol{O}^{Re} \cup \boldsymbol{I}^{re} = \emptyset$ *then* $\boldsymbol{Re}$ *is identified with* $\boldsymbol{U}^{Re}$*,*

and

b. *if* $\boldsymbol{F}^{Re} \cup \boldsymbol{R}^{Re} \cup \boldsymbol{O}^{Re} \cup \boldsymbol{I}^{Re} \neq \emptyset$ *then any* $x \in \boldsymbol{U}^{Re}$ *satisfies one of the following conditions: it is an element of a certain set* $F \in \boldsymbol{F}^{Re}$*, it is an argument of a certain relation* $r \in \boldsymbol{R}^{Re}$*, it is an argument or value of a certain function* $o \in \boldsymbol{O}^{Re}$*, it is an element of the set* $\boldsymbol{I}^{Re}$*. Then the system* $\boldsymbol{Re}$ *is called* ***reality****, the set* $\boldsymbol{U}^{Re}$ *is called the* ***universe****, and elements of successive constituents of reality:* ***things****,* ***features****,* ***relations****,* ***operations****,* ***individuals****, respectively.*

Systems of reality are differentiated. They consist of various subsytems, e.g. physical reality includes a microcosmos, a macrocosmos, and many other subsystems.

Definiton 1.b

A reality $\boldsymbol{Re}' = \langle \boldsymbol{U}^{Re'}, \boldsymbol{F}^{Re'}, \boldsymbol{R}^{Re'}, \boldsymbol{O}^{Re'}, \boldsymbol{I}^{Re'} \rangle$ *is called* ***a subsystem of reality*** *Re iff*

a) $\boldsymbol{U}^{Re'} \subseteq \boldsymbol{U}^{Re}$*,*
b) For each feature $F' \in \boldsymbol{F}^{Re'}$ *there is such a feature* $F \in \boldsymbol{F}^{Re}$*, that* $F' \subseteq F$*,*
c) For any relation $r' \in \boldsymbol{R}^{Re'}$ *there is such a relation* $r \in \boldsymbol{R}^{R}$*, that* $r' \subseteq r$*,*
d) For any operation $o' \in \boldsymbol{O}^{Re'}$ *there is such an operation* $o \in \boldsymbol{O}^{Re}$*, that* $o' \subseteq o$*,*
e) $\boldsymbol{I}^{Re'} \subseteq \boldsymbol{I}^{Re}$*.*

The set of all subsytems of a given reality ***Re*** *we denote by* ***Sub(Re)****.*

In any reality, things can be distinguished in various ways as existing or they can co-exist in varied relations, i.e. things can be in various states: distinction or co-existence in relations. In the first case the description of the state is equivalent to pointing out these things, and in the second, to pointing out n-tuples of things connected with each other.

Definition 2.

A state of things *is any n-tuple* $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ *of elements* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *of the universe* U*, where*

- *for* $n = 1$ $\langle \alpha_1 \rangle = \alpha_1$*,*
- $\langle \alpha_1, \alpha_2, \ldots, \langle \beta_1, \beta_2, \ldots, \beta_k \rangle, \ldots, \alpha_n \rangle = \langle \alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots, \beta_k, \ldots, \alpha_n \rangle$.

Every set of states of things is called a ***structure of states of things:*** *briefly: a* ***structure****.*

The basic state of things, pointing to its manner of existence in a given reality, is the state of co-existing with other things which possess defined features, are in defined relations, are subject to suitable operations, or are distinguished as individuals within this reality. Such states of things create a **structure** (generalized graph) in the given reality, in which these states are directed ways connecting things in a defined order, by means of relations and operations. The set of all such states of things is called the **structure of reality**.

Definition 3.

Let $\boldsymbol{Re} = \langle \boldsymbol{U}^{Re}, \boldsymbol{F}^{Re}, \boldsymbol{R}^{Re}, \boldsymbol{O}^{Re}, \boldsymbol{I}^{Re} \rangle$ *be a reality.* ***Structure of reality*** *it called the set* S^{Re} *of states of things such that:*

- *if* $\boldsymbol{F}^{Re} \cup \boldsymbol{R}^{Re} \cup \boldsymbol{O}^{Re} \cup \boldsymbol{I}^{re} = \emptyset$ *then* $\boldsymbol{S}^{Re} = \boldsymbol{U}^{Re}$*,*
- *if* $\boldsymbol{F}^{Re} \cup \boldsymbol{R}^{Re} \cup \boldsymbol{O}^{Re} \cup \boldsymbol{I}^{re} \neq \emptyset$ *then*

(R1) $\emptyset \notin \boldsymbol{S}^{Re}$*,*

(R2) $\boldsymbol{I}^{Re} \subseteq \boldsymbol{S}^{Re}$*,*

(R3) $\bigcup \boldsymbol{F}^{Re} \subseteq \boldsymbol{S}^{Re}$*,*

(R4) $\bigcup \boldsymbol{R}^{Re} \subseteq \boldsymbol{S}^{re}$,

(R5) $\bigcup \boldsymbol{O}^{re} \subseteq \boldsymbol{S}^{Re}$,

(R5) *for any operation* $o \in \boldsymbol{O}^{Re}$, *if* $\langle \beta_1, \beta_2, \ldots, \beta_k, \alpha \rangle \in o$, *then* $\langle \beta_1, \beta_2, \ldots, \beta_k \rangle \in \boldsymbol{S}^{Re}$ *(the* ***input state of operation****) and* $\alpha \in \boldsymbol{S}^{Re}$ *(the* ***output state of operation****), where* $\alpha = o(\beta_1, \beta_2, \ldots, \beta_k)$,

(R6) *any state of things* $\alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in \boldsymbol{S}^{Re}$, *if for every* $1 \leq i \leq n$, *there are such indices* i_1 *and* i_2, *that* $1 \leq i_1 \leq i \leq i_2 \leq n$ *and for some relation* $r \in \boldsymbol{R}^{Re}$, *or operation* $o \in \boldsymbol{O}^{Re}$: $\langle \alpha_{i_1}, \alpha_{i_1+1}, \ldots, \alpha_{i_2} \rangle \in r$ or $\langle \alpha_{i_1}, \alpha_{i_1+1}, \ldots, \alpha_{i_2} \rangle \in o$, *respectively,*

(R7) *no other state of things except that which satisfied conditions (R2)-(R6) belongs to* $\boldsymbol{S}^R$

All states of things can, but do not have to, belong to the structure of reality. Moreover, a given structure of reality can correspond to various realities. Unfortunately, we don't know the conditions of unambiguity, defining when a given structure of reality can be unambiguously ascribed to a certain reality.

3.2. System of knowledge

In the process of cognition our aim is that our knowledge (base of knowledge), usually expressed in language (e.g. natural language), be **satisfied** in **cognitive reality** by indicating the states of things corresponding to this knowledge which there are in this reality (indicating the **cognitive structures**). Verifying whether knowledge is satisfied in cognitive reality expands our knowledge to **metaknowledge** (**metabase**), which we usually express in metalanguage, employing simultaneously the same apparatus for the transforming of information, i.e. analogous rules of inference.

The above-mentioned theoretical understanding of the notion of satisfaction for such bases of knowledge as mathematical theory was first given by Tarski [1933]. Below we present a generalization of this notion. In this paper we will not, however, consider the criteria of the validity of cognitive processes because these considerations do not have an influence on the formalization of a definition of truth.

Let $\boldsymbol{CR} = \langle \boldsymbol{U}^{CR}, \boldsymbol{F}^{CR}, \boldsymbol{R}^{CR}, \boldsymbol{O}^{CR}, \boldsymbol{I}^{CR} \rangle$ be a reality, called the **cognitive reality**, and $\boldsymbol{BK} = \langle \boldsymbol{U}^{BK}, \boldsymbol{F}^{BK}, \boldsymbol{R}^{BK}, \boldsymbol{O}^{BK}, \boldsymbol{I}^{BK} \rangle$ be a reality called the **base of knowledge** and its components be called, respectively:

- $\boldsymbol{U}^{BK}$ – the **domain of knowledge**, its elements – **units of knowledge**,
- $\boldsymbol{F}^{BK}$ – the family of **categories of terms**; elements of these categories are called **general terms**,

- $\boldsymbol{R}^{BK}$ – the set of **rules**,
- $\boldsymbol{O}^{BK}$ – the set of **functors**,
- $\boldsymbol{I}^{BK}$ – the set of **individual terms**.

Let also every unit of knowledge be an argument or the value of a functor. Elements of the domain of knowledge $\boldsymbol{U}^{BK}$, which are not terms are called **formulas**. Terms which are defined by means of other functors and terms (their **arguments**) are called **complex terms**, while those that are not are called **simple terms**. Formulas created by means of functors and terms (their **arguments**) are called **atomic formulas**. Formulas which are not atomic formulas are called **complex formulas**. If f is a functor and its arguments are terms $t_1, t_2, \ldots, t_n$ then the formula determined by f and these terms is denoted by $f(t_1, t_2, \ldots, t_n)$.

The relation $\vdash_{BK}$ will be called the **relation of derivation of knowledge** in the base $\boldsymbol{BK}$ iff

($\boldsymbol{BK}$) For any set $X \subseteq \boldsymbol{U}^{BK}$ and any $\alpha \in \boldsymbol{U}^{BK}$,

$$X \vdash_{BK} \alpha \text{ if and only if}$$

there exists a sequence $\alpha_1, \alpha_2, \ldots, \alpha_n$ of elements of the domain $\boldsymbol{U}^{BK}$ such that $\alpha_n = \alpha$ and for each $i \leq n$ one of the following conditions is satisfied:

- $\alpha_i \in X$,
- α_i is an individual term,
- α_i belongs to a certain category of general terms $K \in \boldsymbol{F}^{BK}$,
- there is a finite subsequence $\beta_1, \beta_2, \ldots, \beta_k$ of the mentioned sequence and a rule $r \in \boldsymbol{R}^{BK}$ or a functor $o \in \boldsymbol{O}^{BK}$ such that $\langle \beta_1, \beta_2, \ldots, \beta_k, \alpha_i \rangle \in r$ or $\langle \beta_1, \beta_2, \ldots, \beta_k, \alpha_i \rangle \in o$.

The expression $X \vdash_{BK} \alpha$ is read: the unit of knowledge α is derivable from the set X in the base $\boldsymbol{BK}$. In particular, for $X = \emptyset$, it is read: the unit of knowledgeα is a **findings** in this base.

Let us denote by **Val** the set of partial functions v: $\boldsymbol{U}^{BK} \rightarrow \boldsymbol{S}^{CR}$, mapping units of knowledge of the domain of knowledge $\boldsymbol{U}^{BK}$ into states of things belonging to the structure $\boldsymbol{S}^{CR}$ of the cognitive reality $\boldsymbol{CR}$ such that

- for any $t \in \boldsymbol{I}^{BK}$ there is such an individual $i \in \boldsymbol{I}^{CR}$, that for every function $v \in$ **Val**, $v(t) = i$,
- for any category $K \in \boldsymbol{F}^{BK}$ and $t \in K$ there is such a feature $F \in \boldsymbol{F}^{CR}$, that for every function $v \in$ **Val**, $v(t) \in F$.

These functions are called **valuations**.

Let us denote a couple $\langle \alpha, v \rangle$, for any $\alpha \in \boldsymbol{U}^{BK}$ and $v \in$ **Val**, by $\alpha[v]$. The expression $\alpha[v]$ is read: a unit of knowledge α satisfied in the cognitive reality $\boldsymbol{CR}$ by a valuation v, or a unit of knowledge α satisfied in the cognitive $\boldsymbol{CR}$ by ascribing to the unit of knowledge α the state of things $v(\alpha)$.

Let $\boldsymbol{MBK} = \langle \boldsymbol{U}^{MBK}, \boldsymbol{S}^{MBK}, \boldsymbol{R}^{MBK}, \boldsymbol{O}^{MBK}, \boldsymbol{I}^{MBK} \rangle$ be such an expansion of $\boldsymbol{BK}$ to a new base that

1. $\boldsymbol{U}^{MBK} = \boldsymbol{U}^{BK} \cup \boldsymbol{U}^{BK} \times \mathbf{Val}$,
2. $\boldsymbol{F}^{MBK} = \boldsymbol{F}^{BK} \cup \{F|$ there exists such $X \in \boldsymbol{F}^{BK}$, that $F = X \times \mathbf{Val}\}$,
3. $\boldsymbol{I}^{MBK} = \boldsymbol{I}^{BK} \cup \boldsymbol{I}^{BK} \times \mathbf{Val}$,
4. For any rule $x \in \boldsymbol{R}^{BK}$ there is only one such rule $y \in \boldsymbol{R}^{MBK}$, that $x = y|\boldsymbol{U}^{BK}$.
5. For any rule $y \in \boldsymbol{R}^{MBK}$ there is only one such rule $x \in \boldsymbol{R}^{BK}$, that $x = y|\boldsymbol{U}^{BK}$.
6. For any valuation $v \in \mathbf{Val}$, any rule $r \in \boldsymbol{R}^{MBK}$ and every n-tuple $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$:
$$\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in r|\boldsymbol{U}^{BK} \text{ iff } \langle \alpha_1[v], \alpha_2[v], \ldots, \alpha_n[v] \rangle \in r,$$
7. For any functor $x \in \boldsymbol{O}^{BK}$ there is only one such functor $y \in \boldsymbol{O}^{MBK}$ that $x = y|\boldsymbol{U}^{BK}$,
8. For any functor $y \in \boldsymbol{O}^{MBK}$ there is only one such functor $x \in \boldsymbol{O}^{BK}$ that $x = y|\boldsymbol{U}^{BK}$,
9. For any valuation $v \in \mathbf{Val}$, any functor $o \in \boldsymbol{O}^{MBK}$ and every n-tuple $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$:
$$\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in o|\boldsymbol{U}^{BK} \text{ iff } \langle \alpha_1[v], \alpha_2[v], \ldots, \alpha_n[v] \rangle \in o.$$

The system $\boldsymbol{MBK} = \langle \boldsymbol{U}^{MBK}, \boldsymbol{F}^{MBK}, \boldsymbol{R}^{MBK}, \boldsymbol{O}^{MBK}, \boldsymbol{I}^{MBK} \rangle$ is called **a metabase of knowledge**, and elements of $\boldsymbol{U}^{MBK}$ are called **units of metaknowledge**. The relation of derivation in the metabase is denoted by $\vdash_{MBK}$; The expression $X \vdash_{MBK} \alpha$ is read: the unit metaknowledge α is derivable from the set X in the metabase $\boldsymbol{MBK}$. If $X = \emptyset$ this expression is read: the unit of metaknowledge α is a **finding** in this metabase.

We assume henceforth, analogously to Tarski's theory of truth, that knowledge about satisfying complex formulas derives from the metabase on the ground of knowledge about satisfying simple terms and atomic formulas. Such an understanding of the notion of saisfaction can be defined inductively in the following way:

Let
$$In: \boldsymbol{F}^{MBK} \cup \boldsymbol{O}^{MBK} \cup \boldsymbol{I}^{MBK} \to \boldsymbol{F}^{CR} \cup \boldsymbol{R}^{CR} \cup \boldsymbol{O}^{CR} \cup \boldsymbol{I}^{CR},$$
be a one-to-one function, called the **function of interpretation of the base of knowledge $\boldsymbol{BK}$ in the cognitive reality $\boldsymbol{CR}$**, such that

a) For any individual term $t \in \boldsymbol{I}^{MBK}$
$$in(t) \in \boldsymbol{I}^{CR},$$
and for every valuation v:
$$\emptyset \vdash_{MBK} t[v] \text{ iff } v(t) = in(t),$$

b) For any category $K \in \boldsymbol{F}^{MBK}$:
$$in(K) \in \boldsymbol{F}^{CR}$$
and for any general term $t \in K$ and any valuation v:
$$\emptyset \vdash_{MBK} t[v] \text{ iff } v(t) \in in(K),$$

c) For any functor $f \in \boldsymbol{O}^{MBK}$, whose arguments are terms $t_1, t_2, \ldots, t_n \in \boldsymbol{U}^{BK}$ and $f(t_1, t_2, \ldots, t_n) \in \boldsymbol{U}^{BK}$,
$$o = in(f) \in \boldsymbol{O}^{CR}$$
and for any valuation v:
$$\emptyset \vdash_{MBK} f(t_1, t_2, \ldots, t_n)[v] \text{ iff } v(f(t_1, t_2, \ldots, t_n)) = o(v(t_1), v(t_2), \ldots, v(t_n)),$$

d) For any atomic formula $A(t_1, t_2, \ldots, t_n)$, where $A \in \boldsymbol{O}^{MBK}$ is a formula-forming functor and $t_1, t_2, \ldots, t_n$ are terms,
$$r = in(A) \in \boldsymbol{R}^{CR}$$
and for any valuation v:
$$\emptyset \vdash_{MBK} A(t_1, t_2, \ldots, t_n)[v] \text{ iff } \langle v(t_1), v(t_2), \ldots, v(t_n) \rangle \in r,$$

e) Let for a valuation v, S be a set of units of knowledge $k \in \boldsymbol{U}^{BK}$, such that
$$\emptyset \vdash_{MBK} k[v],$$
then for every complex term $t \in \boldsymbol{U}^{BK}$
$$\{k[v] | k \in S\} \vdash_{MBK} t[v] \text{ iff } \emptyset \vdash_{MBK} t[v].$$

f) Let for a valuation v, S be a set of units of knowledge $k \in \boldsymbol{U}^{BK}$, such that
$$\emptyset \vdash_{MBK} k[v],$$
then for every complex formula $\alpha \in \boldsymbol{U}^{BK}$
$$\{k[v] | k \in S\} \vdash_{MBK} \alpha[v] \text{ iff } \emptyset \vdash_{MBK} \alpha[v].$$

Definition 4.

If there exists a function of interpretation of the base of knowledge $\boldsymbol{BK}$ *in the cognitive reality* $\boldsymbol{CR}$*, then for this function*

a. *The set* $\boldsymbol{CS}$ *of states of things for the cognitive reality* $\boldsymbol{CR}$ *defined by formula:*
$$\boldsymbol{CS} = \{v(k) | k \in \boldsymbol{U}^{BK}, v \in \mathbf{Val} \text{ and } \emptyset \vdash_{MBK} k[v]\}$$

is called the ***cognitive structure****. That structure is the set of all states of things assigned by the valuation v to all those units of knowledge that are satisfied by these valuations in cognitive reality (i.e. that are findings in the metabase* $\boldsymbol{MBK}$*) and*

b. *The ordered system:*

$$\boldsymbol{SK} = \langle \boldsymbol{CR}, \boldsymbol{BK}, \boldsymbol{CS} \rangle$$

is called the ***system of knowledge****.*

c. *A term (resp. a formula)* $k \in \boldsymbol{U}^{BK}$*, such that there is* $v \in \mathbf{Val}$ *and* $v(k) \in \boldsymbol{S}^{CR}$*, is called a* ***finding-term*** *for the valuation v.*

d. *By the* ***evaluation*** *we mean the mapping* $ev : \boldsymbol{CS} \to \{\boldsymbol{CR}, \emptyset\}$*, which assigns the cognitive reality* $\boldsymbol{CR}$ *to a state of things s of the cognitive structure* $\boldsymbol{CS}$ *if there exists such a valuation v that for any unit of knowledge k, for which* $v(k) = s$*, k is a finding for valuation v and the unit of metaknowledge* $k[v]$ *is derivable in the metabase from some simple terms and atomic formulas arising from the expansion to metaknowledge of some findings (simple terms and atomic formulas) for valuation v, otherwise* $ev(s) = \emptyset$*.*

e. *The composition of the valuation v and the evaluation ev is called* ***satisfaction*** *for the valuation v.*

f. *The* ***unit of knowledge k is satisfied for the valuation*** *v, if the satisfaction for the valuation v assigns a cognitive reality to it.*

g. *The* ***unit of knowledge k takes parts in the system of knowledge SK****, if it is satisfied for a valuation v.*

From Definition 4 immediately follows:

Theorem 1.

a. *For any functor* $f \in \boldsymbol{O}^{MBK}$ *and its arguments* $k_1, k_2, \ldots, k_n \in \boldsymbol{U}^{BK}$*, for any valuation v:*

$$\emptyset \vdash_{MBK} f(k_1, k_2, \ldots, k_n)[v] \quad \text{iff} \quad \emptyset \vdash_{MBK} f(k_1[v], k_2[v], \ldots, k_n[v]),$$

b. *For any rule* $r \in \boldsymbol{R}^{MBK}$ *and for any units of knowledge* $k_1, k_2, \ldots, k_n \in \boldsymbol{U}^{BK}$*, that the rule r operates, and for any valuation v:*

$$r(k_1, k_2, \ldots, k_n) \quad \text{iff} \quad r(k_1[v], k_2[v], \ldots, k_n[v]).$$

c. *For any valuation v and any* $k \in \boldsymbol{U}^{BK}$*,*

$$\text{if } \emptyset \vdash_{BK} k \text{ then } \emptyset \vdash_{MBK} k[v].$$

Theorem 1 can be used when we want to show that the complex formula is satisfied for a given valuation.

3.3. System of truth

In the process of knowledge it can turn out that **truth** is such that one of the following situations take place:

- our knowledge corresponds to objective reality,
- our knowledge is internally contradictory,
- our knowledge corresponds only to a cognitive reality,
- states of things assigned by cognition do not correspond to those that take place in objective reality.

In the process of cognition of truth we simultaneously get to know how we participate in it as well as how we become embroiled in various relations between the **objective reality**, the **system of knowledge** and the **structure of objective reality**, i.e. **states of things** that really are. We also gain knowledge about conditions (**situational contexts**) in which arise knowledge corresponding to subsystems of objective reality that are accessible to cognition. Then, knowledge is a **reflection of objective reality**. For example, a scientific experiment carried out in suitable conditions (situational contexts) leads us to gain knowledge which corresponds to the objective reality.

The remarks given above justify to a certain degree the introduction of the following definitions:

Definition 5.

The ***system of truth*** *or* ***truth*** *we call a triad* $\boldsymbol{ST} = \langle \boldsymbol{OR}, \boldsymbol{SK}, \boldsymbol{OS} \rangle$*, where* $\boldsymbol{OR}$ *is a reality called the* ***objective reality****,* $\boldsymbol{SK}$ *is the system of knowledge,* $\boldsymbol{OS}$ *is the structure of objective reality.*

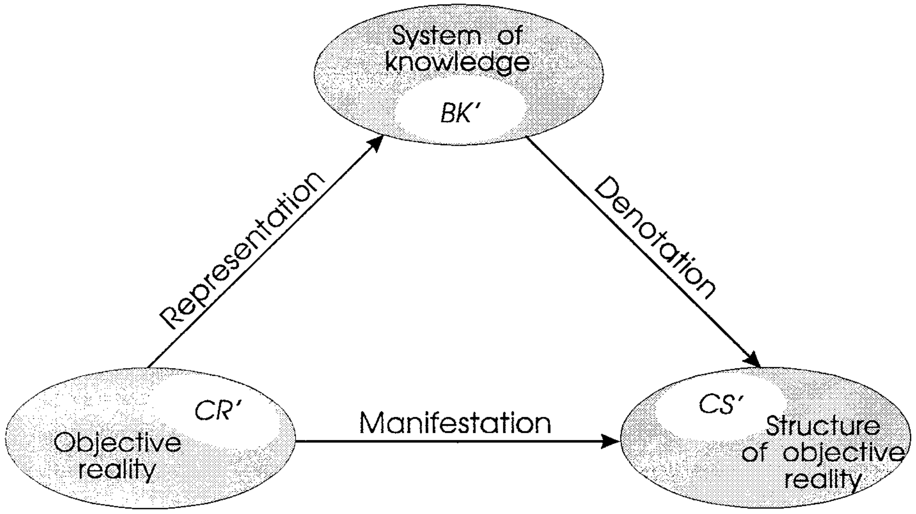
Definition 6.

Let $\boldsymbol{SK}' = \langle \boldsymbol{CR}', \boldsymbol{BK}', \boldsymbol{CS}' \rangle$ *be such a system of knowledge for a given system of truth* $\boldsymbol{ST} = \langle \boldsymbol{OR}, \langle \boldsymbol{CR}, \boldsymbol{BK}, \boldsymbol{CS} \rangle, \boldsymbol{OS} \rangle$ *that* $\boldsymbol{CR}' \in \boldsymbol{Sub(OR)}$ *and* $\boldsymbol{BK}' \in \boldsymbol{Sub(BK)}$*. Then*

A. *The subsystem* $\boldsymbol{CR}'$ *of the objective reality* $\boldsymbol{OR}$ *we call the* ***cognitable subsystem of the objective reality*** *and the subsystem* $\boldsymbol{BK}'$ *of the base of knowledge* $\boldsymbol{BK}$ *we call the* ***reflection of objective reality****.*

B. *The interpretation function s for the system of knowledge* $\boldsymbol{SK}'$ *we call a* ***situational context****, and any valuation in* $\boldsymbol{SK}'$ *restricted to terms we call the* ***reference****. We say about a unit of knowledge k, belonging to the domain of the situational context s, that it is satisfied by the* ***context*** *s* ***in the objective reality****, if the knowledge k is satisfied by every valuation v in* $\boldsymbol{CR}'$ *for the context s.*

C. *Any partial functions re i de such that the first assigns reflections of* *the objective reality to cognitable subsystems of the obective reality* $\boldsymbol{CR'}$ *and the second assigns cognitive structures to reflections of objective reality, in such a way that system* $\langle \boldsymbol{CR'}, re(\boldsymbol{CR'}), (re(\boldsymbol{CR'}))\rangle$ *is a system of knowledge, are called respectively: the* ***representation of objective reality in the base of knowledge*** *(briefly:* ***the representation****) and the* ***denotation****, while the composition of these functions we call* ***the manifestation of the objective reality*** *(briefly: the* ***manifestation****; see Diagram 3),*



Manifestation(CR') = Denotation(Representation(CR'))

The system $\langle \boldsymbol{CR'}, \boldsymbol{BK'}, \boldsymbol{CS'}\rangle$ ***is a system of knowledge***

Diagram 3

D. A partial function which assigns: to any unit of knowledge k, belonging to the base of knowledge, the set of all situational contexts in which the given unit is satisfied in the objective reality, is called the ***denotation*** *of k.*

4. The notion of truthfulness

In the theory of truth presented here, a very important definition is the definition of the truthfulness of units of knowledge. It allows us to interpret

most ways of understanding the concepts of truth and truthfulness that are well-know in the literature. It is sufficient to answer the question: In what truth and in what manner, in the given understanding, do units of knowledge take part?

Definition 7.

Units of knowledge take part in the truth*, if for a certain system of truth they are findings in the system of knowledge or for a certain situational context they satisfy one of the following conditions: they represent objective reality, they possess a denotation, they inform how objective reality manifests in this context.*

Definition 8.

Units of knowledge are true *if they take part in a system of truth.*

As is known, **syntax** describing language employs only terms of various types of language expressions, including those expressions in different categories, for example, the category of expressions that have the same construction (i.e. syntactic categories). In natural language these terms occur as the phrases: "expression...", "sentence...", "conjunction of sentence.... and ...." etc. In this understanding of a base of knowledge about language there is no place for formulas in the sense introduced in this paper. **Semantics**, however, is such a base of knowledge in which some units of knowledge expressed in language cannot be explained by referring to language categories but only within the framework of rules of inference or by referring them directly to states of things that are in a cognitive reality.

Definition 9.

a. ***Truth is semantic*** *if formulas belong to its domain of knowledge (although terms can also belong to it).*
b. ***Truth is syntactic*** *if formulas do not belong to its domain of knowledge.*
c. ***Units of knowledge are semantically true*** *if they take part in a semantic truth.*
d. **Units of knowledge are syntactically true** *if they take part in a syntactic truth.*

In practice, we often use syntactic truth and semantic truth in an equivalent way. The below-given theorem explains why it is permitted.

Theorem 2.

Units of knowledge are semantically true if and only if they are syntactically true.

Proof.

Let a unit of knowledge k be semantically true in the system of truth $\boldsymbol{ST}$. Let us transform the cognitive reality of this system in the following way:

- let us add to its universe the structure of cognitive reality,
- let us add to its family of features the set of relations,
- let us add to the family of features such a family of sets of states of things that each set is ascribed to a functor which is not a furmula-forming functor and conversely, to this set belongs only these states of things which are not values of a valuation for which a formula built by means of this functor is satisfied,
- let us expand the set of operations replacing every relation by an identity operation on n-tuples which are in this relation,
- let us expand the set of operations by such operations that each of them corresponds to a functor f which is not a formala-forming functor and, conversely, in accordance with a formula:

$$o = \{\langle k_1[v], k_2[v], \ldots, k_n[v], k[v]\rangle | k = f(k_1, k_2, \ldots, k_n) \ \ and \ \ \emptyset \vdash_{MBK} f(k_1, k_2, \ldots, k_n)[v]\}.$$

Let us create in the base of knowledge new categories of terms in such a way that to each newly created category belong only previously true formulas formed by means of the same functor: atomic or complex ones. Then we obtain the system of knowledge in which there are no formulas and a unit of knowledge k is satisfied for a valuation if it was earlier satisfied. Thus, the unit k is syntactically true in the new system obtained in this way.

Conversely, if a unit of knowledge k is syntactically true in the system $\boldsymbol{ST}$ then in the cognitive reality it is sufficient to assume that the universe is the earlier indicated structure of cognitive reality and to assume that sets of features, relations and individuals are empty sets. In the base of knowledge it is necessary to assume that the family of categories of terms is empty and the set of individual terms is empty too. We receive then a system of semantic truth in which the set of terms is empty and the unit of knowledge k is syntactically true.

Let us observe that in the second part of the proof of Theorem 1 the definition of semantic truth is such that unit of knowledge k expressed in language by expression "e" is true if and only if that which this expression says is directly identified with the state of things represented by "e", i.e.

expression "e" is true iff e.

So, from Theorem 2 follows the correctness of Tarski's Convention (T).

5. Truth and language

Let us recall that at the beginning we defined the system of truth, briefly truth, as a triad composed of

⟨an objective reality, a system of knowledge and a structure of objective reality⟩.

Because of the narrow framework of this presentation we will limit our considerations to the truth in a logical sense (cf. Carnap 1949, Hintikka 1966). What is truth in a logical sense? We will explain it.

First, we will identify the objective reality with the cognitive reality ($\boldsymbol{OR} = \boldsymbol{CR}$) that enables us to identify this reality with a reality $\boldsymbol{OR} = \boldsymbol{CR} = \boldsymbol{R} = \langle \boldsymbol{U}^R, \boldsymbol{F}^R, \boldsymbol{R}^R, \boldsymbol{O}^R, \boldsymbol{I}^R\rangle$ and $\boldsymbol{S}^{CR} = \boldsymbol{S}^R = CS$ (see Definitions 3 and 4).

Therefore the system of truth is understood as the following triad:

⟨cognitive reality, a system of knowledge and a cognitive structure⟩

Symbolically,

$$\langle \boldsymbol{R}, \boldsymbol{SK}, \boldsymbol{S}^R\rangle.$$

If the system $\boldsymbol{SK}$ of knowledge, i.e. the universe $\boldsymbol{U}^{BK}$ of the base of knowledge $\boldsymbol{BK}$ is expressed in a certain language, natural or artificial (e.g. computer), i.e $\boldsymbol{SK}$ is a language system of knowledge $\boldsymbol{LSK}$, then

Definition 10.

The system of truth is a triad $\langle \boldsymbol{R}, \boldsymbol{LSK}, \boldsymbol{S}^R\rangle$ and will be called the ***truth in the logical sense.***

Let us observe that in this paper the notion of satisfaction of units of knowledge is defined in such a way that for knowledge expressed by formalised language in a certain system of truth in the logical sense, it is reducible to the standard of satisfaction originated by Tarski and used thusfar in formal logic. In truth, variable are then terms, function symbols denote complex term-forming functors, predicate symbols create atomic formulas, propositional connectives and quantifiers (here are sufficient: negation, implication, the universal quantifier) are complex formula-forming functors, while such rules of inference as detachment, substitution, joining and omission of quantifiers are rules of the base of knowledge. Such construction of formalised language for a given cognitive reality (relational system) is such that there is a situational context common for all valuations determined by a given system of knowledge. Hence, in a metabase of knowledge, for derivation of knowledge about the satisfaction of formulas in cognitive reality for a given valuation, knowlege about how these formulas are satisfied by suitable satisfaction of variables occuring in these formulas is sufficient. So the notion of satisfaction in the given situational context can be replaced by a standard one. Thus the following theorem holds:

Theorem 3.
The notion of satisfaction of formulas in a cognitive reality on the ground of the system of knowledge expressed in formalised language can, in the system corresponding to semantic truth in the logical sense, be replaced by the notion of satisfaction in Tarski's sense.

Let the system of truth in the logical sense be the system of syntactic truth considered for categorial languages. The base of knowledge in this system will have the set of categories consisting of only syntactic categories or one category of all terms interpreted as representing states of things which are. However, the set of rules is empty. Then the valuations defining the satisfaction of units of knowledge in cognitive reality (called the ontological structure of categorial language) can be interpreted as a denotation operation in Wybraniec-Skardowska's sense (1998). In this understanding, valuations are homomorphisms of the base of knowledge into cognitive reality (Wybraniec-Skardowska, Rogalski, 1999), to which syntactically true units of knowledge ascribe states of things belonging to the set $\boldsymbol{T}$ of states of things that are ($\boldsymbol{T}$ is simultaneously a feature "being a state of things that is" and a constituent of the ontological structure of categorial language. So,

Theorem 4.
Units of knowledge syntactically true in Wybraniec-Skardowska's sense are true in the logical sense.

Let us observe that the language system of knowledge ***LSK*** has as its base of knowledge a ***language domain of knowledge***, whose units of knowledge are usual expressions: individual terms, general terms or sentence-formulas. Usually that language is understood as a categorial language. We recall that a categorial language is understood as a language in which all well-formed expressions are divided into syntactic categories; such a language can be generated by a so-called categorial grammar.

According to Peirce's well-known distinction *type-token*, the system ***LSK*** can be built at the two levels: at the *token-level* and at the *type-level* (cf. Wybraniec-Skardowska (1985, 1989, 1991) and Rogalski and Wybraniec-Skardowska (1998, 1999)).

Let us consider an exemplification of a ***natural language domain of knowledge*** of which the units of knowledge are expressions of a categorial fragment of natural language, analyzed both as the language of the *token*-expressions, i.e. physical, material linguistic objects (at the *token*-level) and the language of the *type*-expressions, i.e. the classes of token-expressions, abstract, ideal entities (at the *type*-level).

Then we see that the term ***situational context*** is any partial function which assigns *token*-expressions of the language to respective elements of reality $\boldsymbol{R}$ in such a way that:

- relative to proper names are the elements of the set $\boldsymbol{I}^R$;
- to universal names (predicate names) – the features from the set $\boldsymbol{F}^R$;
- to name-forming functors – the functions from the set $\boldsymbol{O}^R$; and
- to some sentence-forming functors – the relations from the set $\boldsymbol{R}^R$.

In the above-mentioned way, according to intuition, the expressions *represent* elements of reality by the situational context. So, the set of all situational contexts creates a representation of reality. We will denote it by $\boldsymbol{S}$.

5. Truthfulness of units of knowledge

For introducing a definition of semantic truthfulness we need the notions of a *function of reference* and a *concrete realization.*

The term ***function of reference***, briefly ***reference*** o in the context s belonging to the representation $\boldsymbol{S}$, will be used for any mapping that assigns any *token*-name to an object from the universe $\boldsymbol{U}^R$ of reality in such a way that

$$o(c) = s(c), \text{ if } c \text{ is a proper name,}$$

$$o(n) \text{ belongs to } s(n), \text{ if } n \text{ is a universal name.}$$

For instance a mapping, by use of a demonstrative pronoun "this ..." that maps the name "house" to a house distinguished in reality, is a reference.

An expression containing a universal name that is not preceded by a quantifier word does not represent any state of affairs if it is in no situational context. For example, the expression "a house has its entrance from street-side" represents a certain state of things in the following contexts: "this house", "the house I mean", etc.

Any mapping r which assigns *representatives* (*token*-expressions) of *type*-expressions ***LSK*** to *type*-expressions ***LSK*** will be called the ***concrete realization*** of the *type*-signs in a system of knowledge. If E is a *type*-expression and r is a concrete realization of a *token*-expression, then the *token*-expression which is a representative of E at r we will denote by $E[r]$.

The expression "$\boldsymbol{R} \models_s E[v]$" is read: a *type*-expression E is true in the reality $\boldsymbol{R}$ for the valuation v in the context s. This expression we define as follows:

Definition 11.
$\boldsymbol{R} \models_s E[v]$ *iff for a concrete realization r there is such a reference o that the valuation v is the composition ($v = r * o$) of the concrete realization r and the reference o and that representative $E[r]$ of the type-expression E is satisfied in the reality $\boldsymbol{R}$, in the situational context s for valuation v.*

Let us now introduce the semantic definition of truthfulness

Definition 12. *(of semantic truthfulness)*
The type-expression E of a language taking part in truth (in the logical sense) is

- ***true in the logical sense in the situational context*** *s for valuation v, if $\boldsymbol{R} \models_s E[v]$,*
- ***true in the logical sense for valuation*** *v (symbolically: $\boldsymbol{R} \models E[v]$), if for any situational context s, $\boldsymbol{R} \models_s E[v]$,*
- ***true in the logical sense in the situational context*** *s (symbolically: $\boldsymbol{R} \models_s E$), if for any valuation v, $\boldsymbol{R} \models_s E[v]$,*
- ***true in the logical sense*** *(symbolically: $\boldsymbol{R} \models E$), if for any situational context s and any valuation v, $\boldsymbol{R} \models_s E[v]$.*

Let us begin with the syntactic definition of the truthfulness of a *type*-sentence (cf. Wybraniec-Skardowska, 1998):

Definition 13. *(of syntactically true)*
For any type-sentence E and for any concrete realization r if for every token-sentence $e \in E$,

$$e \textit{ iff } E[r],$$

then

$$E \textit{ is syntactically true iff } E[r].$$

Thus, a type-sentence E is syntactically true iff for any concrete realization r the representative $E[r]$ corresponds to a state of things that is (in the reality $\boldsymbol{R}$), because we assume that the appearance in a language domain of knowledge of the element $E[r]$ of class E means that it corresponds to a state of things that is in the reality $\boldsymbol{R}$, for any situational context s and any valuation $v = r * o$, where "o" is a reference, i.e. the type-expression E is satisfied in the reality $\boldsymbol{R}$, in the situational context s for valuation $v = r * o$.

Let us observe that this definition in Tarski (1933) was only a definitional postulate (the convention (T)) which the definition of truth should satisfy.

Within the formal theory of truth postulated by the authors, it can be proved that:

Theorem 5.
For any type-sentence E:

$$E \textit{ is syntactically true iff } E \textit{ is true in the logical sense } (\boldsymbol{R} \models E)$$

i.e. the type-sentence E is syntactically true iff it is semantically true, i.e. it has a model in $\boldsymbol{R}$.
Proof.

Let E be syntactically true. Because by Definition 13 we assume that the appearance in a language domain of knowledge of the element $E[r]$ of class E means that it corresponds to a state of things that is in the reality $\boldsymbol{R}$, for any situational context s and any valuation $v = r * o$, where "o" is a reference, then we have $\boldsymbol{R} \models E$.

Let $\boldsymbol{R} \models E$. Then, for any situational context s and any valuation v, $\boldsymbol{R} \models_s E[v]$. In view of Definition 13, for a realization r, $E[r]$.

We will also introduce the notion of denotation d of type-expressions (see Definition 7) in such a way that denotation d is a mapping that ascribes *type*-sentences to sets of all situational contexts in which those sentences are satisfied, for any valuation, in the reality $\boldsymbol{R}$, i.e. to certain subsets of the set $\boldsymbol{S}$ of all situational contexts.

We will say that: an expression A **represents** reality $\boldsymbol{R}$, when $d(A) = \boldsymbol{S}$.

We will prove that: only expressions representing reality $\boldsymbol{R}$ are names representing the features of existing objects of reality as well as true sentences.

And from that by Definiton 12 we will infer that:

Theorem 6. *(on denotation)*

If E is a type-sentence, then E is true iff $d(E) = S$,

If E is a type-sentence, then sentence E is true (syntactically and also semantically) iff denotation of this sentence is the set S of all situational contexts, where d is function od denotation.

7. Perspectives for research

- The definition of truth in the logical sense given in this paper allows us to understand why, when we locate a system of knowledge corresponding to that truth in another system of truth, e.g. corresponding to certain empirical theory, then true units of knowledge in the one system of knowledge can cease to be true in the another system. For example, the sentence "every person is the user of a market or not" is true in the logical sense but does not correspond to truth in the system of knowledge of economics.
- The theory of truth presented here also allows us
  1) to correctly formulate problems connected with finding criteria for:
     - the validity of truth and the rules of derivability of knowledge.
     - the correspondence of knowledge to objective reality. and
  2) to positively solve these problems on the ground of the theory.

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STUDIES IN LOGIC, GRAMMAR AND RHETORIC 3 (16) 1999

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THEOREMS PROVING IN A CERTAIN SEQUENT CALCULUS

1. It seems that Gentzen-like systems are used first of all as techniques for proving of tautologies of a given logic. The techniques in many cases lend themselves to automation. In a less degree the systems are applied to prove theorems in a given theory. Presented here calculus (see also [6]) bases on a suitable Gentzen-type axiomatization and uses the notion of a tree of sequents as a basis tool. It allows to prove not only programs properties expressed in one of logics of programs (for example: Hoare logic ([3]), algorithmic logic ([4]) or dynamic logic ([2])) but also algorithmic properties of data structures (such that dictionaries, natural numbers or stacks). Described here system is presented for algorithmic logic but can be applied (with some modifications) for others logics of programs. Some of its decomposition rules simulate metainduction reasoning. The system only for propositional part of algorithmic logic (in the part formulas and programs can be build out of propositional and program variables and logical and program connectives) is sound and complete but the class of formulas provable in it can be enlarged by additional rules.

2. The algorithmic logic is an extension of the first-order logic by the expressions of one of the following forms:

$$M\alpha \quad \text{or} \quad \bigcup M\alpha \quad \text{or} \quad \bigcap M\alpha$$

where:

$\bigcup, \bigcap$ are iteration quantifiers and
M is a program:

- *assignment statement:* $(x := \tau)$ or $(q := \gamma)$ where x is an individual variable, τ is a classical term, γ is an open formula and q is a propositional variable;
- *composed program:* **begin** $M; M'$ **end**;

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- *branching program:* **if** γ **then** M **else** M' **fi**;
- *iteration program:* **while** γ **do** M **od** where γ is an open formula and M and M' are programs.

The informal meaning of the formula $M\alpha$ is „after execution of the program M the formula α holds" and the informal meaning of iteration quantifiers is following

$$\bigcup M\alpha = \alpha \vee M\alpha \vee M(M\alpha) \vee M(M(M\alpha)) \vee \ldots$$

$$\bigcap M\alpha = \alpha \wedge M\alpha \wedge M(M\alpha) \wedge M(M(M\alpha)) \wedge \ldots.$$

3. Usually in Gentzen-like systems each of the decomposition rules describes relation between its conclusion (written over a line) and its premise or premises (written under the line):

$$\frac{\text{conclusion}}{\text{premise1; premise2; ... premiseN}}$$

Below there are two examples of such rules (the main rules of the sequent calculus under discussion are presented in APPENDIX):

$$\frac{\{\Gamma, (\alpha \wedge \beta), \Delta\}}{\{\Gamma, \Delta, \alpha\}\ ;\ \{\Gamma, \Delta, \beta\}} \qquad \frac{\{\Gamma, (\alpha \vee \beta), \Delta\}}{\{\Gamma, \Delta, \alpha, \beta\}}$$

where Γ, Δ are sequents, i.e. finite sequence (maybe empty) of formulas.

But the standard deduction systems for programs logics include infinitary (so not implemented) rules of inferences i.e. ω-rules:

$$\frac{\text{conclusion}}{\text{premise1; premise2; ... premiseN; premise(N+1); ...}}$$

An example of the ω-rule in algorithmic logic is following:

$$\frac{\{\Gamma, \neg\mathbf{while}\ \gamma\ \mathbf{do}\ M\ \mathbf{od}\alpha, \Delta\}}{\{\Gamma, \Delta, \neg(\mathbf{if}\ \gamma\ \mathbf{then}\ M\ \mathbf{fi})^i(\neg\gamma \wedge \alpha)\}_{i \in N}}$$

Because the most of formal proofs about programs, particularly proofs in a given data structures, are in reality induction proofs then in the system mentioned above rules are replaced by *metainduction*. It allows in the proof process to prove only that the proof exists instead of carrying out full proof for a given formula. The notion of metainduction can be formalized

by Gentzen-type systems in this way that the conclusion and premises of the rules are presented as ordered pair of the form

$$<\Pi, \mathcal{A}>$$

where Π denotes main sequent and $\mathcal{A}$ stands for set (maybe empty) of sequents that are called *metainduction assumption.* The notion of validity of the main sequent Π of the ordered pair with respect to $\mathcal{A}$ is defined in the following way: Π *is valid assuming that each sequent from the set* $\mathcal{A}$ *is also valid.* In our calculus ω-rules are replaced by *metainduction rules.* The metainduction rule is, for example, following:

$$\frac{<\{\Gamma, \neg\mathbf{while}\ \gamma\ \mathbf{do}\ M\ \mathbf{od}\alpha, \Delta\},\ \mathcal{A}>}{<\{\Gamma, \Delta, \neg\delta\},\ \mathcal{A}>\ ;\ <\{\Gamma, \Delta, \neg\ \mathrm{IF}(\mathrm{IF}^j\delta)\},\ \mathcal{A} \cup \{\{\Gamma, \Delta, \neg\mathrm{IF}^j\delta\}\}>}$$

where

$\delta = \neg\gamma \wedge \alpha$

IF = **if** γ **then** M **fi**

j is a parameter of natural type.

This rule presents the metainduction resoning: the left premise of the rule is the metainduction initial condition and the right one corresponds with the metainduction step.

4. In consequence of realization of this idea the standard notion of sequent and process of inferences are modified in the present system.

The main sequent Π of the ordered pair $<\Pi, \mathcal{A}>$ is said to be

- *indecomposable* iff no rule can be applied to it;
- *fundamental* iff the formulas α and $\neg\alpha$ belong to the sequent Π
- $\mathcal{A}$*–provable* iff there exists a sequent $\Sigma \in \mathcal{A}$ such as $\Sigma \subseteq \Pi$ (we will say sometimes that Π is $\mathcal{A}$*–provable with respect to the sequent* Σ);
- A**–provable* iff there exists a sequent $\Sigma \in \mathcal{A}$ for which at least one formula $\beta \in \Sigma$ is with negation and there exists a program M such as

  $$\forall_{\alpha \in \Sigma} \exists_{\beta \in \Pi} (\beta = \alpha_M)$$

  where $\alpha_M = \pm M\alpha'$ if $\alpha = \pm\alpha'$ and $\pm \in \{\neg, \varepsilon\}$ (we will say sometimes that Π is $\mathcal{A}^*$*–provable with respect to the sequent* Σ *and the program* M);
- *terminal* iff Π is indecomposable but Π is neither fundamental nor $\mathcal{A}$–provable nor $\mathcal{A}^*$–provable.

A *proof* of the sequent Π is a diagram (diagram is a decomposition tree obtaining by application of decomosition rules to the input formula) of the sequent such that all paths of the diagram are finite and each its leaf is labelled by the ordered pair $<\Pi, \mathcal{A}>$ where Π is fundamental or $\mathcal{A}$-provable or $\mathcal{A}^*$-provable.

According to the above definition the main sequent of the ordered pair

$$< \{\neg p, \neg K\textbf{if } p \textbf{ then } K \textbf{ fi}^j(\neg p \wedge q), q_1, Kq_1, KK \bigcup Kq_1\},$$
$$\mathcal{A} = \{\{\neg\textbf{if } p \textbf{ then } K \textbf{ fi}^j(\neg p \wedge q), q_1, K \bigcup Kq_1\}\} >$$

is $\mathcal{A}^*$-provable because for $\Sigma = \{\neg\textbf{if } p \textbf{ then } K \textbf{ fi}^j(\neg p \wedge q), q_1, K \bigcup Kq_1\}$ and for $\mathbf{L} = K$ the condition of the definition holds for each $\alpha \in \Sigma$.

Example. Let us consider the following ordered pair:

$$< \{\neg\gamma \rightarrow (\alpha \equiv \textbf{while } \gamma \textbf{ do } M \textbf{ od } \alpha\}, \emptyset > .$$

In the proof process after application rules for logical connectives: $\rightarrow$ and $\equiv$ we obtain two sequents:

(1) $< \{\gamma, \neg\alpha, \textbf{while } \gamma \textbf{ do } M \textbf{ od } \alpha\}, \emptyset >$

(2) $< \{\gamma, \neg \textbf{ while } \gamma \textbf{ do } M \textbf{ od } \alpha, \ \alpha\}, \emptyset >$.

To the first of them we apply: first the rule for program connective **while – do – od** and next the rule for logical connective $\wedge$. In this way we obtain two fundamental sequents:

(1.1) $< \{\gamma, \neg\alpha, \neg\gamma, \gamma \wedge \ M(\textbf{while } \gamma \textbf{ do } M \textbf{ od } \alpha)\}, \emptyset >$

(1.2) $< \{\gamma, \neg\alpha, \alpha, \gamma \wedge \ M(\textbf{while } \gamma \textbf{ do } M \textbf{ od } \alpha)\}, \emptyset >$.

To the second of them we can apply the metainduction rule obtaining two sequents:

(2.1) $< \{\gamma, \neg(\neg\gamma \wedge \alpha), \alpha\}, \emptyset >$

(2.2) $< \{\gamma, \neg \textbf{ if } \gamma \textbf{ then } M \textbf{ fi}(\textbf{ if } \gamma \textbf{ then } M \textbf{ fi})^i\ (\neg\gamma \wedge \alpha), \alpha\},$
$\{\gamma, \neg\ (\textbf{if } \gamma \textbf{ then } M \textbf{ fi})^i\ (\neg\gamma \wedge \alpha), \alpha\} >$.

From the (2.1) sequent (after application of the rule for negation of conjunction) we obtain the fundamental one:

(2.1.1) $< \{\gamma, \neg\neg\gamma, \neg\alpha, \alpha\}, \emptyset >$.

From the (2.2) sequent (after application of the rule for negation of program connective **if – then – fi**) we obtain two sequents:

(2.2.1) $< \{\gamma, \neg\gamma, \neg M(\textbf{ if } \gamma \textbf{ then } M \textbf{ fi})^i\ (\neg\gamma \wedge \alpha), \alpha\},$
$\{\gamma, \neg\ (\textbf{if } \gamma \textbf{ then } M \textbf{ fi})^i\ (\neg\gamma \wedge \alpha), \alpha\} >$

(2.2.2) $< \{\gamma, \gamma, \neg(\textbf{ if } \gamma \textbf{ then } M \textbf{ fi})^i\ (\neg\gamma \wedge \alpha), \alpha\},$
$\{\gamma, \neg\ (\textbf{if } \gamma \textbf{ then } M \textbf{ fi})^i\ (\neg\gamma \wedge \alpha), \alpha\} >$.

The first of them is fundamental and the second of them is $\mathcal{A}$–provable.

5. In this way we obtain the proof system for algorithmic logic that allows to prove validity of formulas in automatic way suitable for computer realization.

Main facts about the system:

- the system is sound and complete for propositional part of this logic;
- the system is decidabled for propositional algorithmic logic;
- any valid formula of propositional algorithmic logic becomes a valid formula of the first-order algorithmic logic following a substitution of formulas for propositional variables and programs for programs variables;
- the system extended to the first-order part of the logic is not complete (and it must be not complete because the first-order algorithmic logic is not even partial decidabled);
- the class of formulas provable in the system for the first-order algorithmic logic can be enlarged by some modifications and other special rules.

6. The main problems connected with automated developing proofs in a given theory (the proof process in this situation are more complcated) are following:

- finding appropriate scheme of metainduction;
- extension of the notion of the $\mathcal{A}^*$-provable sequent;
- finding appropriate terms to substitute for the individual variables when the rules for classical quantifiers are applied.

6.1. Sometimes it can appear that proposed earlie metainduction scheme is not enough good and presented below one is proper:

$$\frac{< \{\Gamma, \neg\textbf{while } \gamma \textbf{ do } M \textbf{ od}\alpha, \Delta\},\ \mathcal{A} >}{< \{\Gamma, \Delta, \neg\delta\},\ \mathcal{A} > ;\ < \{\Gamma, \Delta, \neg \mathrm{IF}\delta\},\ \mathcal{A} > ;\ <\{\Gamma, \Delta, \neg\ \mathrm{IF}(\mathrm{IF}(\mathrm{IF}^j\delta))\},\ \mathcal{A} \cup \{\{\Gamma, \Delta, \neg \mathrm{IF}^j\delta\}, \{\Gamma, \Delta, \neg\ \mathrm{IF}(\mathrm{IF}^j\delta)\}\} >}$$

where

$\delta = \neg\gamma \wedge \alpha$

$\mathrm{IF} = \textbf{if } \gamma \textbf{ then } M \textbf{ fi}$.

This rule presents the metainduction reasoning where the first two premises of the rule are the metainduction initial conditions and the third premise corresponds with the metainduction step.

6.2. The class of provable formulas can be enlarged by extension of the notion of the $\mathcal{A}^*$-provable sequent to the notion of the $\mathcal{A}^{**}$-provable sequent. The main sequent Π of the ordered pair is said to be

> A^{**}-*provable* iff there exists a sequent $\Sigma \in \mathcal{A}$ for which at least one formula $\beta \in \Sigma$ is with negation and there exists a program M such as
>
> $$\forall_{\alpha\in\Sigma}\exists_{\beta\in\Pi}(\alpha_M \Rightarrow \beta)$$
>
> where $\alpha_M = \pm M\alpha'$ if $\alpha = \pm\alpha'$ and $\pm \in \{\neg, \varepsilon\}$

6.3. The problem of finding appropriate terms to substitute for the individual variables (when the rules for classical quantifiers (Appendix: rule (12) and (13)) are applied) is very important during automated theorem proving. One way to overcome this problem is to introduce unification. The other problem connected with the unification of terms it is the problem of removing classical quantifiers from formulas. Skolemising is one of the well known method that allows to remove existencial quantifiers and to introduces constants or proper functions. It appears that for algorithmic formulas so also for the formula of the type

$$\forall_x \exists_y ((y := \varphi(y))P(x,y))$$

the task is not so easy. In the described here system an alternative approach to Skolemising is used as very suitable for logic of programs. The main idea of the mechanism ([1], [5]) is following:

- first, the tree ordering of the formula, denoted by $<$, is introduced; the relation describes dependence between terms and variables which have been introduced during process of building the diagram of the formula,
- next, during the process of unification for each substitution σ(which substitutes terms for variables) a certain relation $< \bullet$ is introduced with some restriction: the transitive closure of the union of the relation $<$ and the relation $< \bullet$ does not lead to a cycle.

Now the proof process with unification will be described more precisely. First a notion of algorithmic formula in partial prefix form will be defined as more suitable for automated theorem proving.

Definition Algorithmic formula is in partial prefix form if and only if it is in the following form

$$\mathbf{Q}_1 x_1 \mathbf{Q}_2 x_2 \ldots \mathbf{Q}_n x_n \alpha$$

where

- $\mathbf{Q}_i x_i \in \{\forall, \exists\}$ dla $i \in \{1, \ldots, n\}$,
- in the formula α only subformulas $M\beta$, $\bigcup M\beta$ i $\bigcap M\beta$ can include classical quantifiers.

The steps of the proving with unification are following:

Step 1. Turn the input algorithmic formula into partial prefix form performing the following tasks:

- eliminate logical connectives $\Leftrightarrow$ and $\Rightarrow$ using below equivalences

  $\alpha \Leftrightarrow \beta$ is equivalent $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$,

  $\alpha \Rightarrow \beta$ is equivalent $\neg\alpha \vee \beta$

- move logical connective $\neg$ to inside a formula using below equivalences

  $\neg\neg\alpha$ is equivalent α

  $\neg(\alpha \vee \beta)$ is equivalent $\neg\alpha \wedge \neg\beta$

  $\neg(\alpha \wedge \beta)$ is equivalent $\neg\alpha \vee \neg\beta$

  $\neg\forall_x\alpha$ is equivalent $\exists_x\neg\alpha$

  $\neg\exists_x\alpha$ is equivalent $\forall_x\neg\alpha$

- change the variables: if two quantifiers have the same bound variable x (for example), then change in one of the quantifiers the variable x into y and change all free occurrence of x in the range of the quantifier. Example: the formula $\forall_x(P(x) \vee \exists_x R(x,x))$ is changed into the formula $\forall_x(P(x) \vee \exists_y R(y,y))$.
- move classical quantifiers on the beginning of a given formula using below equivalences, where $\mathbf{Q} \in \{\forall, \exists\}$ and R is without the variable x:

  $\mathbf{Q}xP(x) \vee R$ is equivalent $\mathbf{Q}x(P(x) \vee R)$

  $\mathbf{Q}xP(x) \wedge R$ is equivalent $\mathbf{Q}x(P(x) \wedge R)$

  $\forall_x P(x) \wedge \forall_x R(x)$ is equivalent $\forall_x(P(x) \wedge R(x))$

  $\exists_x P(x) \vee \exists_x R(x)$ is equivalent $\exists_x(P(x) \vee R(x))$

  $\mathbf{Q}_1 xP(x) \vee \mathbf{Q}_2 xR(x)$ is equivalent $\mathbf{Q}_1 x\mathbf{Q}_2 z(P(x) \vee R(z))$

  $\mathbf{Q}_1 xP(x) \wedge \mathbf{Q}_2 xR(x)$ is equivalent $\mathbf{Q}_1 x\mathbf{Q}_2 z(P(x) \wedge R(z))$

Step 2. During applying the rules for classical quantifiers build unification diagram for a given decomposition tree of the formula. The unification diagram is the ordered pair $<\mathcal{D} \cup \mathcal{Z}, \succ>$, where

- $\mathcal{D}$ is a set of variables (called d-variables), that are some kind of imitations (on this level of proving we do not know which terms will be proper) of terms that are introduced by rules for existential quantifiers;
- $\mathcal{Z}$ is a set of variables (called z-variables), that are introduced by rules for general quantifiers;
- $\succ$ is a relation such as for arbitrary variables $t_1, t_2 \in \mathcal{D} \cup \mathcal{Z}$ it holds that $t_1 \succ t_2$ if and only if the quantifier $\mathbf{Q}_2$ introducing the variable t_2 is in the immediate range of the quantifier $\mathbf{Q}_1$ introducing the variable t_1.

Step 3. During checking whether a given sequent Π is fundamental or not, if only the sequent is not fundamental and the set $\mathcal{D} \cup \mathcal{Z} \neq \emptyset$, do the unification in the following way:

- look for pair of formulas $\alpha, \beta \in \Pi$ such that

$$\alpha = Id\varrho(\tau_1, \ldots, \tau_n) \quad \text{and} \quad \beta = \neg Id\varrho(\tau'_1, \ldots, \tau'_n),$$

  where $\tau_1, \ldots, \tau_n, \tau'_1, \ldots, \tau'_n$ are terms and ϱ is n-ary predicat,
- find such substitution σ, that
  - $\sigma(\tau_1) = \sigma(\tau'_1), \quad \ldots \quad , \quad \sigma(\tau_n) = \sigma(\tau'_n)$ and
  - after adding to the unification diagram for the substitution σ two relations: equivalence one $\sim$ and additional one $\ll$ that are defined in the following way:
    - ◇ if $\sigma = [d_1 \setminus d_2]$, then $d_1 \sim d_2$,
    - ◇ if $\sigma = [d_1 \setminus \tau]$ and τ is not d-variable then $d_2 \ll d_1$ and $z \ll d_1$ for each d-variable d_2 occurring in τ and for each variable z occurring in τ;
    - ◇ $\sim$ satisfies the conditions of equivalence relation and if $d_2 \sim d_1$ i $z \ll d_1$ then also $z \ll d_2$,

the transitive closure of the union of the relation $\succ$ and the relation $\ll$ does not lead to a cycle.

Example. Let us consider the following formula

$$\exists_y (F(y) \Rightarrow \forall_x F(x)).$$

The partial prefix form of the formula is following:

$$\exists_y \forall_x (\neg F(y) \vee F(x)).$$

In the second step of the proof process with unification we build the unification diagram for the input formula introducing the relation

$$d_1 \succ z_1$$

After application of the decomposition rules for the formula we obtain the sequent with the formulas

$$\neg IdF(d_1),\ IdF(z_1),\ \exists_y \forall_x (\neg F(y) \vee F(x)).$$

In the next step of the proof process with unification we check whether the sequent is fundamental. This sequent will be fundamental if we do the unification introducing the substitution $\sigma = [d_1 \setminus z_1]$. This substitution is permissible if the following adding dependence

$$(*) \qquad z_1 \ll d_1$$

does not lead to a cycle in transitive closure of the union of the relation $\succ$ and the relation $\ll$. In our case the substitution σ is not permissible and there is no another substitution that does not lead to a cycle. In this situation we can only extend our decomposition tree by applying rules for classical quantifiers. We extend, at the same time, the relation $\succ$ introducing the following dependence

$$d_2 \succ z_2$$

between the new d-variable d_2 and z-variable z_2. Now we obtain the sequent with the following formulas:

$$\neg IdF(d_1),\ IdF(z_1),\ \neg IdF(d_2),\ IdF(z_2),\ \exists_y \forall_x (\neg F(y) \vee F(x)).$$

We check whether the sequent is fundamental. We know that the substitution $\sigma = [d_1 \setminus z_1]$ is not permissible. So we look for other one that enables us to do the unification. Such substitution will be $\sigma = [d_1 \setminus z_2]$ because the following dependence

$$z_2 \ll d_1$$

added to the unification diagram does not lead to a cycle.

7. Presented here ideas of theorems proving in the sequent calculus for algorithmic logic are basis for computer system that has been designed and implemented ([6]). The computer system can support users in proving of programs properties in formal way (in order to carry out a nontrivial proof formally it is necessary to do it step by step justifying each of them by a precise rule of inference but the large number of tedious details often makes an impossible obstacle in carrying out the proof in tradicional manner).

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APPENDIX

DECOMPOSITION RULES FOR SEQUENT CALCULUS

Let Γ denotes a set of indecomposable formulas; Δ is an arbitrary set of formulas; α, β are arbitrary formulas; γ denotes a propositional formula; j is a parameter of natural type; M, M', M'' denote arbitrary programs; $\circ \in \{-,+\}$; $\mp \in \{\neg, \epsilon\}$; $\Box \in \{\wedge, \vee\}$; $Q \in \{\bigcup, \bigcap\}$; $pref^\circ$ is a sequence of simple programs.

Rules for logical connectives:

(1)
$$\frac{<\{\Gamma, pref^\circ \neg\neg\alpha, \Delta\}, \mathcal{A}>}{<\{\Gamma, pref^\circ \alpha, \Delta\}, \mathcal{A}>}$$

(2)
$$\frac{<\{\Gamma, pref^\circ \mp(\alpha \Rightarrow \beta), \Delta\}, \mathcal{A}>}{<\{\Gamma, pref^\circ \mp(\neg\alpha \vee \beta), \Delta\}, \mathcal{A}>}$$

(3)
$$\frac{<\{\Gamma, pref^\circ \mp(\alpha \Box \beta), \Delta\}, \mathcal{A}>}{<\{\Gamma, pref^\circ \mp\alpha, \Delta\}; \{\Gamma, pref^\circ \mp\beta, \Delta\}, \mathcal{A}>}$$

where $(\circ, \mp, \Box) \in \{(-, \epsilon, \vee), (+, \epsilon, \wedge), (+, \neg, \vee), (-, \neg, \wedge)\}$

(4)
$$\frac{<\{\Gamma, pref^\circ \mp(\alpha \Box \beta), \Delta\}, \mathcal{A}>}{<\{\Gamma, pref^\circ \mp\alpha, pref^\circ \mp\beta, \Delta\}, \mathcal{A}>}$$

where $(\circ, \mp, \Box) \in \{(+, \epsilon, \vee), (-, \epsilon, \wedge), (-, \neg, \vee), (+, \neg, \wedge)\}$

Rules for programs connectives and iteration quantifiers:

(5)
$$\frac{<\{\Gamma, pref^\circ s \mp \alpha, \Delta\}, \mathcal{A}>}{<\{\Gamma, pref^\circ \mp s\alpha, \Delta\}, \mathcal{A}>}$$

(6)
$$\frac{<\{\Gamma, pref^\circ \mp s\gamma, \Delta\}, \mathcal{A}>}{<\{pref^\circ \mp s\gamma, \Gamma, pref^\circ \mp \overline{s\gamma}, \Delta\}, \mathcal{A}>}$$

(7)
$$\frac{<\{\Gamma, pref^\circ \mp \mathbf{begin}\ M'; M''\ \mathbf{end}\alpha, \Delta\}, \mathcal{A}>}{<\{\Gamma, pref^\circ \mp \mathbf{if}\ \gamma\ \mathbf{then}\ M'\ \mathbf{else}\ M''\ \mathbf{fi}\alpha, \Delta\}, \mathcal{A}>}$$

(8)
$$\frac{<\{\Gamma, pref^\circ \mp \mathbf{if}\ \gamma\ \mathbf{then}\ M'\ \mathbf{else}\ M''\ \mathbf{fi}\alpha, \Delta\}, \mathcal{A}>}{<\{\Gamma, pref^\circ \mp((\gamma \wedge M'\alpha) \vee (\neg\gamma \wedge M''\alpha)), \Delta\}, \mathcal{A}>}$$

(9)
$$\frac{<\{\Gamma, pref^\circ \mp \mathbf{while}\ \gamma\ \mathbf{do}\ M\ \mathbf{od}\alpha, \Delta\}, \mathcal{A}>}{<\{\Gamma, pref^\circ \mp \bigcup\ \mathbf{if}\ \gamma\ \mathbf{then}\ M\ \mathbf{fi}\ (\neg\gamma \wedge \alpha), \Delta\}, \mathcal{A}>}$$

(10)
$$\frac{<\{\Gamma, pref^\circ \mp Q M\alpha, \Delta\}, \mathcal{A}>}{<\{\Gamma, pref^\circ \mp\alpha, \Delta, pref^\circ \mp M Q M\alpha\}, \mathcal{A}>}$$

where $(\circ, \mp, Q) \in \{(+, \epsilon, \bigcup), (-, \epsilon, \bigcap), (-, \neg, \bigcup), (+, \neg, \bigcap)\}$

(11)
$$\frac{<\{\Gamma, pref^\circ \mp Q M\alpha, \Delta\}, \mathcal{A}>}{<\{\Gamma, pref^\circ \mp\alpha, \Delta\}, \mathcal{A}>;\ <\{\Gamma, \Delta, pref^\circ \mp M(IdM^j\alpha)\}, \mathcal{A} \cup \{\{\Gamma, \Delta, pref^\circ \mp IdM^j\alpha\}\}>}$$

where $(\circ, \mp, Q) \in \{(-, \epsilon, \bigcup), (+, \epsilon, \bigcap), (+, \neg, \bigcup), (-, \neg, \bigcap)\}$

Rules for classical quantifiers:

(12)

$$\frac{<\{\Gamma, \mathit{pref}^{\circ}\mathbf{Q}_x\alpha(x), \Delta\}, \mathcal{A}>}{<\{\Gamma, \mathit{pref}^{\circ}(x := \tau)\alpha(x), \Delta, \mathit{pref}^{\circ}\mathbf{Q}_x\alpha(x)\}, \mathcal{A}>}$$

where $(\circ, \mathbf{Q}) \in \{(+, \exists), (-, \forall)\}$

(13)

$$\frac{<\{\Gamma, \mathit{pref}^{\circ}\mathbf{Q}_x\alpha(x), \Delta\}, \mathcal{A}>}{<\{\Gamma, \mathit{pref}^{\circ}(x := v)\alpha(x), \Delta\}, \mathcal{A}>}$$

where $(\circ, \mathbf{Q}) \in \{(-, \exists), (+, \forall)\}$

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ON NATURAL LANGUAGE PRESENTATION OF FORMAL MATHEMATICAL TEXTS*

The research reports on the presentation of formal mathematical text with a view to their intelligibility to human beings. Input text – article written in **Mizar** language [RudTry99]. Output text is presented in English [Mat84].

Formal proof systems have long been studied as part of mathematical logic, especially proof-checkers systems were originally intended for actual use in carrying out standard mathematical texts. With the advent of powerful and sophisticated implementations of logics, formal reasoning in general, and formal proofs in particular, are becoming relevant and accessible to other fields that use and rely on mathematical reasoning techniques. Though the use of formalisms enables formal proofs to be written by human and checked with the help of the computer, such articles have one serious deficiency: they are overburdened by large amounts of technical detail of the underlying formal systems. This formal view obscures the basic line of reasoning and hinders human comprehension. There is evidently a wide gap, then, between formal texts and conventional mathematical proofs, whose essential purpose, in addition to establishing the truth of propositions, is to provide insight and understanding. One may argue that it is not worthwhile trying to understand a formal proof at all, once it has been machine-checked for correctness. This is certainly the case where proofs are technical and tedious, and fail to offer any insights, and we would be happy to leave the verification of such arguments to the machine. But in other cases, a formal mathematical text, just as its informal counterpart, carries important information that we would like to communicate.

In the research we propose an approach to formal text presentation that attempts to combine formality with comprehensibility. This approach is gu-

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ided by an analogy relating the activities of proving and programming. Although proving in **Mizar** is declarative process, programming is procedural process. Developing a program from a specification is very much like developing a proof for a theorem. By following this analogy, we apply techniques and principles known from program design to proof design and presentation. Most important, we apply the principle of refinement to proofs. Refinement has been used as both an informal and a formal abstraction principle to control complexity and to structure and guide the process of programming. By transferring the refinement paradigm to formal proof design, we will show in the following sections how we arrive at *formal* and *hierarchically structured* texts, which are presented at different levels of abstraction. The upper levels indicate how complete formal text can be constructed, and they carry the essential information that constitutes the basic line of reasoning. The lower levels fill in the necessary technical detail, a task that can be left entirely up to the machine. Thus, the choice of the most appropriate level of abstraction depends of difficulty of the proof, and of those for whom the presentation is intended (education, report of the database, journal, etc.). Its depends also of the mechanical capabilities of the underlying reasoning system, which besides proof-checking of the formal text must also maintain and verify the database of mathematical knowledge.

The article is organized as follows:

- Section 1 describes in general terms our approach to translation of formal language texts into natural language.
- In Section 2 we discuss hierarchically structured formal mathematical text, describes the instantiation of our approach to the case of the **Mizar** language.

1. Overview of machine translation

Systems for translation of formal language require the source text to be expressed within unambiguous grammar or severe syntactic and lexical limits (in comparison to natural language). One of the objectives of such systems is that an author who conforms to the restrictions is rewarded with a reliable and fully automatic translation of the text into one or more target languages (esp. natural ones). Thus a certainty of the *completeness* of such systems is of great importance. A machine translation system is *complete* if and only if all expressions that are correct according to the input language grammar have at least one translation in the output language. The present research is inspired by the method of compositional translation developed

in the Rosetta Project [Ros94]. It focussed on the provability of completeness for relatively simple grammar formalisms, which are more appropriate for machine translation of formal languages. The research reported in this article is motivated primarily by the potential benefit of the use of formal language to the reliability of machine translation.

A formal language is a precisely defined alteration of a part of natural language, on the one hand constrained in its lexicon, grammar and style, on the other hand possibly extended with mathematical specific terminology and grammatical constructions. Formal languages are used for recording mathematical texts which discuss limited and well defined domains, so that the restrictions imposed are more readily accepted. Due to the reduction in ambiguity, redundancy, vocabulary size, and grammatical complexity, texts written in a formal language become easier to analyze and process for computers. Computational processing can benefit from the use of formal language. Natural language is notoriously difficult for computers to process.

By choosing the formal language, it can become possible to guarantee successful computational processing of texts written in that formal language. In the case of machine translation system, adherence to a formal input language could guarantee the successful analysis of any input-language text. That is, we will find conditions under which the machine translation procedure can be guaranteed to be completed. Compositional translation assumes that the intermediary language is defined as compositional grammar, i.e. grammar that obey the well-known compositional principle [*see* ParMeuWal93]. We define a compositional grammar as a finite set of *basic expressions*, a finite set of *syntactic rules*, a finite set of *syntactic categories*, and a syntactic *type assignment function*.

- *basic expressions* are, roughly, the smallest meaningful units in a language (the stems of content words),
- *syntactic rules* are operations that recursively build derived expressions from *basic expressions*. Expressions are either basic or derived expressions,
- *syntactic categories* describe the syntactic properties of expressions,
- basic expressions A all have a *syntactic category(A)*,
- *syntactic rules* restrict their arguments to certain categories, and specify the category of the derived expression they yield,
- the *syntactic type assignment function* associates every *syntactic rule S* with an ordered pair,
- *type assignment function(S)*, consisting of the so-called *argument list(S)*, of the categories of its arguments and its *result category(S)*.

The *arity* of *syntactic rules* is the number of categories in the rule's argument list. We require that *syntactic rules* are total: they must be applicable to any combination of arguments that matches their argument list. According to the compositionality principle the semantic component of a compositional grammar assigns an object from a semantic domain to each basic expression and an operation on object in that domain to each syntactic rule.

We define translation as a relation between expressions, assuming that the translation relation implies equivalence of meaning. Compositional translation is based on the notion of translation equivalence of their syntactic elements. We assume that this translation is specified between their *basic expressions* and their *syntactic rules*, where translation equivalent rules must have the same *arity*.

We will realize compositional translation as follows:

Input (Formal) Language Expression

⇓ **parsing**

IL Syntactic Derivation Trees

⇓ **analysis**

Semantic Deep Structure Trees

⇓ **synthesis**

Output (Natural) Language Derivation Trees

⇓ **generation**

OL Expression

Parsing performs lexical and syntactic analysis of an IL expression, yelding the set of all syntactic derivation trees that correspond to the expression. *Analysis* performs semantical analysis of input formal mathematical text. *Synthesis* of an IL semantic deep structure trees yields the set of all translation-equivalent OL syntactic derivation trees. And the *generation* of a well-formed syntactic derivation tree produces the corresponding OL expression.

2. Hierarchically Structured Mathematical Text

Mathematical text usually has a structure – conventional mathematical language and formal one. Mathematicians communicate with a mixture of natural language and symbols. However the language used is restricted, and certain expressions appear frequently. Conventional mathematical language has never been defined. It is not a formal language in any sense. Proofs do not contain the low-level detail that a formal system or proof-checker would require. Such steps are considered obvious, and hence often omitted, to concentrate on the important steps.

The first observation of the structure of the informal mathematical language is, that the language comes in two variants, which it is useful to distinguish between. Firstly – there is the language for making definitions and their properties. Mathematicians want to be very precise in definitions, and there are idioms which we can rely on the process them, so analysis is expected to be straightforward. Secondly – the language used in proofs – theorem and the stage of the proof provides a rich context in which to interpret sentences. Reasoning can bridge some of gaps, so the user need not be so precise here. At minimum, he/she could give the basic information, which then requires only simple inferences to complete the proof. Thus, the language used is more variable, and more informal, than for definitions.

Next observation, extremely important for our work is, that we can easily divide conventional mathematical text into two parts. One of them is formal one – representing formalized form of text: mathematical ***formulae*** [BanCar93] and standardized phrases. The second one is informal: written in natural language, they convey the authors intuitions, interpretations of some facts and get together formal texts. In other words we will to distinguish two layers: formal layer of text (formalized and standardized) and informal, intuitive layer [TrySwi91]. Andrzej Trybulec in [Try72] calls them: objective layer and subjective layer. This second part of mathematical text we will call in our work ***metatext*** [Mat99a].

Usually mathematical papers are written in a well defined language with relatively poor vocabulary. The language of mathematics mostly consists of:

- a number of standard phrases:
  - definitions:
    - *we define ... by ...*
    - *a ... is ... if ...*
    - *we ... call ... if ...*
  - notations:
    - *we will denote by ...*

let us denote ...
here and subsequently ... denotes ...

- assumptions:
  we will make {need} the following assumptions ...
  it is required {assumed} that ...
  we can certainly assume that ..., since otherwise ...
- theorems:
  the theorem states that ...
  another way of stating ... is to say ...
  we have thus proved ...
- proofs:
  we first prove that ...
  to deduce ... from ... , take ...
  we have divided the proof into a sequence of lemmas ...
  it is plain {clear} {obvious} that ...
  since ... it follows that ...
- a large set of symbols, terms and formulae, with well defined semantics to shorten the text: $\forall, \exists, \neg, \Rightarrow, \Sigma$, etc,
- a number of keywords: *proof, theorem, thus, since, assume, define,* etc,
- a set of geometric figures, graphs of functions, schemas.

All of above shows, that there are possibility to construct standard (formal) language of mathematics. Kałużnin in [Kal62] shown, that language of mathematical logic does not suffice for a standard mathematical papers. He proposed to do:

- Statistical and logical analysis of the language of mathematical texts to show all tendencies for standardization.
- To develop specialized and simplified (in first phase) mathematical language combined with mathematical logic. The fixing of precise rules of the introduction of new terms.

2.1 Formal Mathematical Language

The idea to have mathematical, standardized language is old one. Now with the advances in computer science we have needs to have mathematical texts which will be good for processing by computer. There are two main approaches towards the formalization of proofs to enable automatic verification. One of them there are theorem provers – they interactively seek a proof of a certain theorem, while guaranteeing that the constructed proof is correct. The internal result of such proof is not readable for a human. It is like internal language in programming. Very "close" for computer, "far" for humans. The second approach – proof-checkers are represented by **Mizar** language. Based on classical logic and set theory a language has been developed for writing proofs in the "mathematician's style", ready to check logical correctness by the mechanized verifier.

Let's we observe more general properties of formal mathematical language. It will be important to have general structure of mathematical text for syntactic and semantic analysis. Analyzing formal mathematical texts we can observe its poor vocabulary, the use of standard phrases and keywords that introduce and combine simple sentences, the large use of terms and formulae. Proofs are a highly structured form of discourse. A crucial prerequisite to understand the course of the argumentation within a proof is to identify the discourse relations between sentences of that discourse. Deriving the discourse relations of a given proof means reconstructing the intentional structure and the informational structure of the proof.

Notable representant of the formal mathematical language is **Mizar** language. It is derived from the mathematical vernacular, devised by A. Trybulec [RudTry99]. The structure of the **Mizar** text is as follows:

`environ` *directives* `begin` *text-item* { `begin` *text-item* }

The environment *directives* indicating which items of the Mizar Mathematical Library (MML) can be referenced in the article. *Directives* concerned vocabulary symbols – *predicate, functor, mode, structure, selector, attribute* or *functor bracket.* Library *directives* – notations and constructors, definitions and theorems, clusters and schemes. There are also requirement *directives* – given obviousness of chosen mathematical theory.

Text-item show classical structure of the mathematical text: new definitions and theorems as well as proofs for these. There are also schemas – theorems with second order free variables. Reservations are used to reserve identifiers for a type.

2.2 An Example of Elements of Presentation

Explained above *Text-Item* has proof as one of the items. The structure of the proof is shown in [Mat99b]. Their sentences are presented in English used at least **sorting and ordering** in the process.

Sorting and Ordering – this process reorganizes arguments in the sentence and not reduces number of sentences. Is based on the following heuristic procedures:

a) **connective sentences** (`&`, `or`, `iff`) and symmetric relations (e.g. equality predicate "`=`") fulfill *commutative concept.* After sorting left siding argument have to be shorter sentence or less complex. On the left side can be also a sentence with defined variables. It need to qualify priorities to choice the procedure. Exception holds the *Definiens* in definition.

Let we show one example of sorting for [jordan2b, MatNak97]. Every intermediary step of translation is shown below to observe consecutive difference of natural style:

```
the carrier of TOP-REAL 1 = REAL 1
& REAL 1 = 1-tuples_on REAL;
```

↓ translation without sorting

"*the carrier of* $\mathcal{E}^1_{\mathrm{T}} = \mathcal{R}^1$ *and* $\mathcal{R}^1 = \mathbb{R}^1$."

↓ sorting related to connective "&"

"$\mathcal{R}^1 = \mathbb{R}^1$ *and the carrier of* $\mathcal{E}^1_{\mathrm{T}} = \mathcal{R}^1$."

↓ sorting in the right predicate "="

"$\mathcal{R}^1 = \mathbb{R}^1$ *and* $\mathcal{R}^1$ = *the carrier of* $\mathcal{E}^1_{\mathrm{T}}$."

↓ final output is a consequence of aggregation rule *eliminating logical predicates* explained in [Mat99b]

"$\mathcal{R}^1$ *is equal to* $\mathbb{R}^1$ *and to the carrier of* $\mathcal{E}^1_{\mathrm{T}}$."

b) **to change order of arguments** in *Implication-Sentence*. On the left side we have shorter sentence or less complex, consequently with proper linguistic verbalization:

"*if p, then q*" ⟶ "*q when p*"
"*p implies q*" ⟶ "*q if p*"

c) **to replace argument(s)** of "where" into beginning of formulae, to focus the main object:

```
Ball(c,r) = {q where q is Element of the carrier
             of RealSpace : dist(c,q) < r};
```

↓ translation without sorting

"*Ball(c,r)* = $\{q\colon \rho(c,q) < r\}$,
where q is element of the carrier of the metric space of real numbers."

↓ translation with sorting

"*let q be element of the carrier of the metric space of real numbers in:* *Ball(c,r)* = $\{q\colon \rho(c,q) < r\}$."

d) **itemization of arguments** in *Implication-Sentence*:

```
A & B & C implies D & E & F;
```

↓

"*Suppose that*
(i) A,
(ii) B, and
(iii) C.
Then
(i) D,
(ii) E, and
(iii) F."

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Roman Matuszewski

A STRUCTURAL APPROACH TO HUMAN ORIENTED PRESENTATION OF FORMAL PROOFS*

Traditionally, research on computer aided formalization of mathematics focussed in how the computer can help in the process of constructing correct mechanically-checked mathematical articles. Research consequently concentrated on building library of articles. The biggest, developed since 1989, is Mizar Mathematical Library. The last years have seen a growing interest in reading and using proof-checked articles, by the authors of articles and by the wider audience through the internet. Understanding and then practical application of articles are deeply depend of their presentation. Our approach is based on the idea of D. Knuth [Knu84], which we can paraphrase "Instead of imagining that our main task is to explain to a computer why it is correct, let us concentrate rather on explaining to human beings why it is correct". Such an explanation may use means and devices that are employed in the process of human understanding. Language and notation are used to explain the reasoning, intuition, association and stylistical paraphrasing may be used to help the reader. We therefore propose a planning approach to the presentation of **Mizar** proofs that integrates a formal proof and its explanation into a single document. The support of mathematical notation deserves special attention. Well designed notation, especially in TEX, plays an important role in the communication of mathematical understanding.

Presenting formal mathematical text may be seen as an attractive ides, particularly since formal arguments tend to be bogged down by technical details. These circumstantiality usually hides the basic line of reasoning underlying a proof, but is necessary to enable a computer to check the correctness of an argument. Strictly formal proofs contain too much technical

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detail, which is of no interest to the human reader, who only wishes to understand *the basic idea.* This results in a long, overly detailed proof in which the basic line of reasoning is obscured. The representation of a formal text is geared towards a form that is easy to parse for computer, which differs from the form a human would choose in order to understand it. This results in a lack of structural information. Both above combined result in superfluous information on the one hand and the lack of helpful information on the other hand. But still, such a proof contains a representation of the basic proof idea that was on the mind of the person written the *Mizar article.* Thus, by hiding the unnecessary information and by providing additional information it should be possible to recover the proof idea.

In the case of **Mizar**, the representation of a proof is the collection of the *proof structuring constructs* [RudTry99]:

- *generalization,*
- *assumption,*
- *existential-assumption,*
- *exemplification,*
- *conclusion.*

These *constructs* we can use to recalculate `thesis` of the proof (see details in the Section 2). **Mizar** proofs are in the *spirit* of natural deduction – it permits to reconstruct the reasoning used in ordinary mathematical text and is helpful in providing a global structure locally.

On the other hand, the primary goal of our research should be to convince the user of the correctness of an argument and not the machine. The intelligible presentation of formal proofs is usually not attempted because of their technical detail. We want to separate the discussion of the requirements into two areas: what features are necessary to present formal reasoning in a structured and natural language way, and how can the whole system be kept flexible, i.e., applicable for various instances of formal reasoning. We will kept the discussion at a rather abstract level so that the software architecture of a supports system becomes visible.

Another question is graphical representation of the text. The fact, that the formulas are displayed in the severely restricted `ASCII` character set doesn't add to the comprehensibility either. Thus we take this representation merely as a basis to derive step by step a proof document that is independent from the syntax of the system, well structured, and oriented at common proving styles. Afterwards, a TEX, or `HTML`, document can be generated, where all the operators, constants, and so on are replaced by their appropriate mathematical symbols.

1. A Theoretical Approach to Presenting Formal Proofs

In contrast to the belief that mathematical texts are only schematic and mechanical, *state of the art* techniques of natural language processing are necessary to produce coherent texts that resemble those found in typical mathematical textbooks. The human proof presentation process is based on the natural language generation techniques. In one side we have formal text as input, and as output we will have text in natural language. Traditionally, the generation process has been divided into two stages:

- ***what to say,*** and
- ***how to say it.***

The first stage comprises processing from the concept and intermediate representation of the formal proof to the planning of contents. The second covers realization of the plans into text or output in other modalities.

The planning of the contents we can developed owing to research of Discourse Representation Theory. Realization of the plans – owing to Rhetorical Structure Theory.

a) **Discourse Representation Theory** (DRT) is a theory of the semantics of natural language [GroSid90]:
   - it takes as the *unit of semantic analysis* not the single sentence (and its grammatical constituents) but rather the coherent (possibly multi-sentential) discourse of texts,
   - it maps syntactic structures onto logical forms of discourses and their parts (so-called Discourse Representation Structures – DRS).

These constitute a representation formalism for which a separate semantic is defined which relates DRS and their parts to the subject matter of the discourses they represent. One task in planning (then in generation) is that of constructing or fitting into the discourse structure. This involves:

- selecting the content of the discourse (or part of discourse),
- determining how to segment discourse content,
- finding the relationship between discourse segments and communicative intentions (in our case it is establishing a logical relation between propositions of the proof),
- organizing segments and content so that the discourse is coherent (in our case it is proof plan).

DRT has three distinct but interacting aspects of discourse structure:

- **intentional structure,**
- **attentional structure,**
- **linguistic structure.**

The **intentional structure** might be matched with the purpose in mathematical discourse to fulfill all proof obligations, whereas the **attentional** state, in our domain and approach, might coincide with the currently active proof structure being under refinement. The necessity to perform a multi--level discourse analysis is discussed in [MooPol90]. We agree to the fact that a crucial prerequisite to understand the course of the argumentation within a proof is to identify the discourse relations between sentences of that discourse.

Deriving the discourse relations of a given textbook proof means reconstructing the **intentional structure**:

- describing how sentences within a discourse segment contribute to a common discourse purpose, namely how to decrease the proof obligations

and the **informational structure**:

- describing how sentences within a segment are related to each other by some relation, namely a logical consequence relation

of the proof.

b) **Rhetorical Structure Theory** (RST) is a formalism that describes discourse structure as a set of *rhetorical* relations organized hierarchically [ManTho87]. Originally it was defined for English text, but it has been used in other areas of language processing. The theory includes four kinds of objects:
   - relations,
   - schemas,
   - schema applications,
   - structures.

A relation identifies a relationship that can hold between non-overlapping text pieces. Examples of relations include elaboration, motivation, restatement, contrast and summary. Generally, one of the piece of text is the *nucleus* and others are *satelites*, although some relations are *multi-nuclear* (e.g. sequence, joint). The *nucleus* of the relation is usually the part that is more crucial to the writer's purpose. It is generally coherent on its own; if removed, it might render the text incoherent. In our topic RST does encode some information about intentionality in the "efect" field of the relation definition. In RST text structure is separated from other aspects of text generation, including:

- choice of communicative goals and text content,
- identification of reader,
- theme control,
- lexical choice.

The lexical choice in generation of proof provided differentiation of text – realized by choosing synonymy phrases of natural language mathematical text. In RST-based text planners the RST theory is operationalized by mapping rhetorical relations identified into corresponding plan operators, characterized pieces of text. The global style of a piece of text produced has to be determined by committing to a global rhetorical strategy, since the diversity of the existing proof texts indicates the flexibility of such communicative norms. In other words, the presentation operators represent only the general constraints or strong tendencies motivating to act in a certain way. In many cases, the choice of operators is by no means unique. On the contrary, the choice is strongly influenced by various pragmatic aspects, such as the knowledge level of the addressee, the writer's goal to apply a specific educational strategy, or the goal to emphasize a mathematically interesting part of proof. In other works, such pragmatic aspects of a conversation are first mapped onto a group of rhetorical goals, which are associated with rhetorical strategies. The extra level of rhetorical goals is meaningful, because one rhetorical goal can fulfill different interpersonal goals. A simple treatment is adopted in this theory, where each interpersonal goal is associated directly with a rhetorical strategy, which essentially enforces a partial order on the set of presentation operators in every concrete situation. In other words, in a concrete circumstance the presentation operators are given different priorities by a rhetorical strategy.

Finally, RST is not intended to cover all forms of discourse. One can assume that most discourse (at least English, mathematics) is hierarchically structured and functionally organized, and so can be constructed or analyzed within the RST framework. Implementing RST in an effective fashion has proven somewhat difficult. It is necessary to specify how to find the content of the text, which relations to use in different situations, how many times to use each relation or part of a relation, how to order the *nucleus* and *satellite* in a relation, and how to order and combine the relations themselves [MooPar92].

2. A General View of Formal Proof

We are concerned with construction of formal proofs written in **Mizar** language. Not with syntactical grammar, but with structure. The **Mizar** Syntax we can get from `http://mizar.org/language`. The proof is constructed as follows:

Sentence-to-be-proved `proof` *Reasoning-Items* `end`

Reasoning-Items → *Skeleton-Items*

Reasoning-Items → *Auxiliary-Items*

Five *Skeleton-Items* are the *proof structuring constructs* which form skeleton of the proof and correspondingly modify the value of the formula of the current `thesis`. The keyword `thesis` can be used inside a proof and denotes the yet unproven part of original *Sentence-to-be-proved.*

- *generalization*, `let X be set;` appropriate for `thesis` that starts with a universal quantifier,
- *assumption*, `assume X ∈ A;` appropriate when `thesis` is an implication,
- *existential-assumption*, `given R being Subset of REAL such that X=R;` appropriate when `thesis` is an implication with an existentially quantified antecedent,
- *exemplification*, `take f;` appropriate for thesis starting with an existential quantifier,
- *conclusion*, `thus x ∈ X;` appropriate when thesis can be seen as a conjunction.

Auxiliary-Items are used if necessary, as auxiliary steps in proof. They don't modify the value of the formula of the current `thesis`.

- *per-cases-statement,*
- *choice-statement,*
- *compact-statement,*
- *diffuse-statement,*
- *iterative-equality,*
- *type-changing-statement,*
- *private-definition.*

3. The Basic Structure of the Proof Presentation

The approach can best be described by distinguishing three aspects that influence proof presentation:

- **the high level planning,**
- **the low level planning,**
- **the output realizer.**

The first relates to the global structure of the proof; the second to the sentence planning; and the third to the verbalization of the output.

The global organization of structure of a formal proof has a decisive influence on comprehensibility. The hierarchical structure makes explicit dependencies between the arguments in a proof. **Mizar** proofs emphasize the hierarchical structure of an argument and make its structure explicit in the presentation.

a) **The high level planning** – efforts have been made to profiling [Lan91] **Mizar** proof structure as general concept, for which what is essential is hierarchisation of components. It is called also as content planning of proof. We regard hierarchical structuring as a way to provide the *global* organization of a proof. Intuitively speaking, this level of planning first decides the order in which proof steps should be conveyed. There are also some messages to highlight global proof structure.

The hierarchical planning is realized by top-down presentation steps that split the task of presenting a particular proof into subtasks of presenting subproofs or simple derivation steps. The overall planning mechanism is similar to Rhetorical Structure Theory based planning approach. The output of the high level planning is an ordered sequence of *Proof-Reasoning* actions planned to achieve communicative goals. Like speech acts, these actions can be defined in terms of the communicative goals they fulfill as well as in terms of their possible verbalizations. Similar approach we can find in [Hua94].

Based on an analysis of proofs in mathematical textbooks, there are mainly three types of communicative goals:

- *simple-derivational-steps*, there are *Linkable-Statements*; in terms of rhetorical relations, acts in this category represent a variation of the rhetorical relation concerning derivation,
- *skeleton-steps*, their consists of *Generalization, Assumption, Exemplification* and *Conclusion*; main rhetorical steps in the proof,
- *structural-attentional-steps*, there are: *Sub-Proofs, Diffuse-Statements, Per-Cases*; either convey a partial plan for the forthcoming discourse or signal the beginning or the end of a subproof. Acts of this sort verbalized in English are called *metatext.*

For presenting general structure of proof we use *clue* words, this indicates realization of specific communicative goal.

Metatext are introduced by part of system called Filler [BanCar93]. The translation proceeds to the formulae and back again to Filler. Thus, there is a need in programming to note *initial, medial,* and *final* position of Filler. The translation is strictly controlled, and the objective of creating a natural language result is accuracy while at the same time generating variations (paraphrasing) in the Filler links. Some examples of paraphrasing:

"Let us / "We } first prove that ..."

"We will { make / need } the following assumptions: ..."

"This theorem { deals with ..." / express the equivalence of ..." / states that ..."

"..., which proves the theorem."
"This completes the proof."

Next, we have to say that we refuse detailed labeling every sentence in proof to justify them. This kind of referring needs to change positions of the some reasoning items, and proof can be divided into paragraphs in linear form. This approach is very important, because usually formal texts are in labeled or parentheses structure, which is not natural.

b) The low level planning – the sentence planning comprises, among other subtask, those of combining and reorganizing of linguistic resources associated with functions, predicates, terms, and various types of derivations, in order to produce connected text.

During this stage generated text mirroring the information structure of the proof and the formulae. This means that every step of derivation is translated into a separate sentence, and formulae are recursively verbalized. As an instance of the later the formula from **Mizar** article [jordan2b]:

```
for p being Element of the carrier of TOP-REAL 1
ex r st p=<*r*>;
```

is verbalized as [MatNak97]:

"For every element p of the carrier of $\mathcal{E}^1_{\mathrm{T}}$ there exists r such that p=<r>."

Many of the Natural Language Generation Systems link their information structure to the corresponding linguistic resources either through predefined *templates* or by specific, individual application. Therefore their expressive power is restricted and planning resulted in very mechanical texts. According to our analysis, there are at least 4 linguistic and logical

phenomena that call for appropriate low level planning, increasing fluency of generated text:

- sorting and ordering,
- aggregation,
- logical rewriting based on the logical rules,
- paraphrasing and key words in choosing of justification.

The first phenomenon is explained in [Mat99].

4. Aggregation

Aggregation is the process of removing redundant information during language generation while preserving the information to be conveyed. One of the most important techniques [DalHov93] for sentence planning. Sentence aggregation is the task of grouping sentences (messages) or arguments of sentence, combining two or more messages into one sentence. The aggregation system must decide both what messages to aggregate to form each sentence, and also what syntactic mechanism should be used to combine messages.

Without loss of information, aggregation reorganizes and merges information items at various levels to remove redundancies. Most of rules below can be seen as adaptations and extensions of the grouping rules identified for argumentative texts. A further class of rules is introduced, called *grammatical-grouping* and *reasoning-chain*, as a powerful extension for mathematical text. Altogether, all the rules stated at the discourse and semantic level are integrated. We will show, that the incorporation of aggregation does not change the information content of a text, its does contribute to readability and fluency, as well significantly improves the coherence of text produced. Note, that the aggregation rules work together with sorting strategies (see first example of sorting in [Mat99]).

Types of aggregation

LINGUISTICAL GROUPING RULES – applied to semantic objects as linguistics one, not logical.

On the beginning let we introduce some basic conventions:

CONC(arg1,arg2, ...) – linguistic concatenation of arguments being semantic objects,

for(arg), el(arg1,arg2), bel(arg1,arg2), eq(arg1,arg2), sub(arg1,arg2), sentence – semantic objects,
CONC(arg,arg,arg, ...) = arg
CONC(arg) = arg

a) **simple-conjunction**: the simplest form of aggregation is to use a connective such as *and* to produce a sentence plan which communicates more than one messages. In some cases it may be necessary to add clue words such as *also* or *as well* to increase fluency. It can occur to aggregate only short sentences. Let we take examples from [MatNak97]:

```
f"P=f"(Ball(u,r)); f"(P) is open; f is continuous;
```

↓ CONC(sentence1,sentence2,sentence3)

"$f^{-1}(P)=f^{-1}(Ball(u,r))$ *and* $f^{-1}(P)$ *is open as well f is continuous.*"

b) **relative**: instead conjunction clauses *and* combine sentences using a relative clause: *which, which is, which give us*:

```
x is Element of F; F is Subset of G;
hence x is Element of G;
```

↓ CONC(sentence1,sentence2,sentence3)

"x is an element of F and F is a subset of G, which give us that x is an element of G."

c) **set-formation**: grouping identical syntactical constructions into list of arguments:

`for x for y for z holds . . .` ⟺ CONC(for(x),for(y),for(z))

↓ for(CONC(x,y,z))

"for every x, y, z holds ..."

d) **ellipsis-embedding**: if two messages being aggregated have a common constituent, it may be possible to remove in one of message the repeated constituent. It is mostly realized by textual operation *object-embedding*:

`REAL = the carrier of RealSpace;` ⟺ eq(arg1,arg2)
`u ∈ REAL;` ⟺ bel(arg3,arg1)

↓ CONC(eq(arg1,arg2),bel(arg3,arg1))

"$\mathbb{R}$ *is equal to the carrier of the metric space of real numbers and u belongs to* $\mathbb{R}$."

↓ aggregation bel(arg3,eq(arg1,arg2))

"*u belongs to* $\mathbb{R}$, *which is equal to the carrier of the metric space of real numbers.*"

We can observe a textual decision that no matter how bel and eq should be instantiated, the repeated argument arg1 in bel(arg3,arg1) will be replaced by eq(arg1,arg2). We have to combine eq(arg1,arg2) and bel(arg3,arg1) to form an embedded object bel(arg3,eq(arg1,arg2)). This textual operation eliminates one of the duplicates of arg1.
Another example:

`x is Element of R;` ⟺ el(x,R)
`R is Subset of G;` ⟺ sub(R,G)

↓ CONC(el(x,R),sub(R,G))

"x is an element of R and R is a subset of G."

↓ el(x,sub(R,G))

"x is an element of the subset R of G."

↓ yet one another aggregation – **relative**

"x is an element of R, which is a subset of G."

There is also dual case of embedding when arguments in CONC change their places.

IMPORTANT REMARK: In both above **ellipsis-embedding** examples we can observe, that replacement is realized when replaced object has same type as repeated argument. It means, that we can not insert object el on the first place in the object sub.

e) **eliminating logical predicates**: we can observe very often, that sentences has same predicate with different arguments. In this rule we will eliminate repeated predicates:

`P is Subset R^1;` ⟺ sub(P, R^1)
`Q is Subset TOP-REAL n;` ⟺ sub(Q,TOP-REAL n)

↓ CONC(sub(P, R^1), sub(Q, TOP-REAL n))

"P is a subset of $\mathbb{R}$ *and Q is a subset of* $\mathcal{E}^n_{\mathrm{T}}$*."*

↓ sub(CONC(P,Q),CONC(R^1,TOP-REAL n))

"P and Q are subsets of $\mathbb{R}$ *and* $\mathcal{E}^n_{\mathrm{T}}$*, respectively."*

LOGICAL GROUPING RULE – realized on the semantic objects as logical one, not linguistical.

f) logical-grouping: applying logical tautologies:

`If a then b; If c then b;`

↓

"If a or c then b."

`If a then b; If a then c;`

↓

"If a then b and c."

Specified in shown above rules, our aggregation is more abstract than similar rules reported in the literature. One important feature of our work is the integration of low level planning knowledge specific to our domain of application. This body of knowledge must be refined to further improve the quality of the text produced. More experience is also required to formulate strategies to choose among alternatives.

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RULE COMPLEXES FOR REPRESENTING SOCIAL ACTORS AND INTERACTIONS*

Abstract

In the paper we present the notion of rule complex, several associated concepts, and their elementary properties. The notion of rule complex is a fundamental mathematical concept of the present author and Tom R. Burns'[1] joint research work on reconceptualisation of the social game theory. In our approach both social actors and their interactions (in particular, games) are uniformly represented by rule complexes.

Keywords: social game theory, rule complex.

1. Introduction

Social structures (e.g., organisations, institutions, cultural forms, etc.) and social relationships can be conceptualised as special types of social rule systems [1]. Though one could give many examples of forms of social activity where it would be difficult to identify precisely rules governing these activities, most social actions and activities are based on formal or informal rules. Also most social actors (human or not) are rule-guided.

By a social game we mean a particular form of social interaction where actors involved are conscious of being involved in this interaction. In the classical game theory by von Neumann and Morgenstern [3], games are viewed as collections of pre-determined rules. Players have to follow the

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rules and are not allowed to modify, innovate or transform them unless stated otherwise. However, they may choose a strategy and decide which of the possible moves to take.

In practice, the totality of rules of a social game can have a more complex structure than merely a set of rules. Such a structure can contain, e.g., collective procedures and norms like procedures of order, turntaking or voting in committees and democratic bodies. In the present author and Tom R. Burns' work on reconceptualisation of the social game theory, games are therefore represented by rule complexes which, in particular, may be finite sets of rules. The notion of rule complex, introduced in [2], is a generalisation of (finite) set of rules. The rule complex of a game consists of rules or other rule complexes. It specifies more or less precisely who the actors are, what their roles, rights, and obligations are, what a game it is, what action opportunities, resources, goals, procedures, and payoffs of the game are, etc. We follow the view that rules of a game can be incomplete, imprecise or with exceptions; and that social actors can creatively modify or even dramatically change rules. Indeed, social actors are not merely agents following rules in a strict way or pure rationalists maximising a value. They also try to realise their social relationships and cultural forms, and hence engage in processes of restructuring of games. Thus, *NM* games are particular cases of *BG* games[2].

In our approach actors, rule-guided social beings, are represented by rule complexes. With every social actor we can associate values, norms, actions and action modalities, judgment rules and algorithms, beliefs, knowledge, and roles specific for the actor. They can be represented in the form of rules or rule complexes, and organised into rule complexes. All the rule complexes are composed into an actor's rule complex. Actors apply their rule complexes in situations of action or interaction (in particular, game) to achieve private or group objectives, to plan and implement necessary activities, and to solve problems.

Given a situation S_t, any actor a_i is represented by a rule complex $ACTOR(i,t)$, containing all rules which a_i has in S_t. We call this rule complex an *actor complex* of a_i at t. Some rules of $ACTOR(i,t)$ concern a_i's roles at t. If we neglect (or delete) rules irrelevant for the topic 'role', we shall obtain a subcomplex $ROL(i,t)$ of $ACTOR(i,t)$ called a *role complex* of a_i at t. Suppose that actor a_i has several roles at t, e.g., citizen, employee, wife, and mother. By deletion of irrelevant rules we obtain subcomplexes of

[2] 'NM' stands for 'von Neumann and Morgenstern', while 'BG' for 'Burns and Gomolińska'.

$ROL(i,t)$ specific for particular roles, viz., a citizen-role complex $ROL_c(i,t)$, an employee-role complex $ROL_e(i,t)$, a wife-role complex $ROL_w(i,t)$, and a mother-role complex $ROL_m(i,t)$; all related to the point of time t.

Starting with $ROL(i,t)$, we may want to consider particular aspects like a_i's values, judgment making, norms, modalities of actions and activities, actions or models. By deletion of irrelevant rules for the topic 'values' one may obtain a subcomplex $VAL(i,t)$ of $ROL(i,t)$ called a *value complex* of a_i at t. In a similar way we obtain a_i's *judgment complex* $JUDG(i,t)$, consisting of rules concerning a_i's judgment making at t, a_i's *norm complex* $NORM(i,t)$, consisting of norms which a_i has at t, a_i's *modality complex* $MODALITY(i,t)$, consisting of rules describing action and activity modalities of a_i at t, a_i's *action complex* $ACT(i,t)$, consisting of rules describing possible actions and activities of a_i at t, and a_i's *model* of S_t, $MOD(i,t)$. The subcomplex $MOD(i,t)$ consists of rules representing all explicit beliefs which a_i has about the environment, state of affairs, other actors, and him/herself at t. Proceeding similarly, we obtain subcomplexes of particular role complexes, e.g., of the mother-role complex $ROL_m(i,t)$, concerning values, judgment making, norms, action/activity modalities, actions and activities, and the model, viz., $VAL_m(i,t)$, $JUDG_m(i,t)$, $NORM_m(i,t)$, $MODALITY_m(i,t)$, $ACT_m(i,t)$, and $MOD_m(i,t)$, respectively.

Our framework based on rules and rule complexes enables us to represent in a uniform way both social actors and their interactions, to specify structures and rules of social games in a given context, and to predict the likely actions and interactions among participants, outcomes or developments of a given game.

For any set X, $card(X)$ (resp., $\wp(x)$, and X^2) will denote the cardinality (the power set, and the Cartesian product $X \times X$) of X. $x_0 \in x_1 \in x_2 \in \ldots$ means that $x_0 \in x_1$, $x_1 \in x_2$, etc. Along the standard lines, $\mathcal{N}$ denotes the set of natural numbers.

For lack of space many interesting questions concerning rule complexes as mathematical objects cannot be discussed in the present paper. In Section 2 we consider the notions of rule and rule complex, and associated notions as the notions of rule part, complex part, rule base, complex base, g-element, degree of a rule complex, and subcomplex. In Section 3 we define the notions of derivability of a rule from a rule complex, activation of a rule, and (in)consistency of a rule complex. The problem of application of rules and rule complexes is addressed in Section 4. Section 5 contains a brief summary.

2. Rules and rule complexes

In our approach the concept of social rule is broader than that of norm. We identify and analyse various types of rules and rule systems like norms, laws, moral principles, codes of conduct, rules of the game, administrative regulations and procedures, recipies for action, technical rules, conventions, customs and traditions, category and descriptive rules, evaluative rules or judgment systems [1].

At this stage we do not specify the language of our formalism in detail. We assume that a language $\mathcal{L}$ is given where the object and meta levels may be not separated. Formulae of $\mathcal{L}$ will be denoted by lowercase Greek letters with subscripts whenever necessary. *FOR* will denote the set of all formulae obtained according to some formation rules.

2.1. Rules

Social rules are represented by some triary relations called just rules. Our notion of rule is more general than the traditional concept of inference rule because exceptions are allowed. In fact, our rule is a default rule [4] in the language $\mathcal{L}$. Facts are represented by axiomatic rules, i.e. rules without premises and justifications. Rules will be mainly denoted by r with subscripts if needed. In general, *rul*(x) says that x is a rule.

Formally, a *rule* is a triary relation $r \subseteq (\wp(FOR))^2 \times FOR$ such that for any triples $(X_1, Y_1, \gamma_1), (X_2, Y_2, \gamma_2) \in r$, $card(X_1) = card(X_2) < \aleph_0$ and $card(Y_1) = card(Y_2) < \aleph_0$. Where $(X, Y, \gamma) \in r$, the set X (resp., Y) is a set of *premises* (*justifications*). We assume that premises and justifications do not express orders. γ is called the *conclusion* of r. If $Y = \emptyset$, r is an ordinary *if-then* rule. If $X = Y = \emptyset$, then r is an *axiomatic* rule. Any formula α can be transformed into an axiomatic rule, viz., $r(\alpha) = \{(\emptyset, \emptyset, \alpha)\}$. Some rules are *schematic*[3], i.e., they can be written in the schematic form $r : \frac{X:Y}{\gamma}$. If $X = \{\alpha_0, \ldots, \alpha_m\}$ and $Y = \{\beta_0, \ldots, \beta_n\}$, then we shall write

$$r : \frac{\alpha_0, \ldots, \alpha_m : \beta_0, \ldots, \beta_n}{\gamma} \tag{1}$$

[3] The *modus ponens* rule $MP : \frac{\alpha, \alpha \to \beta}{\beta}$ is an example of a schematic rule.

instead of $r : \frac{\{\alpha_0, \ldots, \alpha_m\}:\{\beta_0, \ldots, \beta_n\}}{\gamma}$. If $X = Y = \emptyset$, r will be written as $r : \frac{}{\gamma}$. $(X, Y, \gamma) \in r$ informally reads as: *If all elements of X hold and all elements of Y may hold, then conclude γ.*

A rule r is a *meta rule* w.r.t. a rule or rule complex x if x is mentioned or used in premises, justifications or the conclusion of r.

Example 2.1. Let $r_0 = \{(\{\alpha_0\}, \emptyset, \gamma_0)\}$ and $r_1 = \{(\{\alpha_1\}, \emptyset, \gamma_1)\}$, where α_0 says: *The premises of r_1 hold* and γ_1 says: *Apply r_0*. The above rules are mutually meta-rules. ■

2.2 Rule complexes

Rule complexes will be mainly denoted by C, D with subscripts whenever needed. *compl*(x) means that x is a rule complex.

Definition 2.2. By a *rule complex* we mean a set obtained according to the following formation rules:

1. Any finite set of rules is a rule complex.
2. If C, D are rule complexes, then $C \cup D$ and $\wp(C)$ are rule complexes.
3. If $C \subseteq D$ and D is a rule complex, then C is a rule complex.

In words, the class of rule complexes contains all finite sets of rules, is closed under the set-theoretical union and the power set, and preserves inclusion[4].

Example 2.3. Let r_1, r_2, r_3, r_4 be rules. Sets $C_1 = \{r_2, r_3\}$, $C_2 = \{r_4\}$, $C_3 = \{r_1, C_1\}$, and $C_4 = \{r_1, C_2, C_3\}$ are rule complexes. ■

Example 2.4. Algorithms as collections of instructions may be seen as rule complexes. ■

Proposition 2.5. For any rule complex C and a set D, $C \cap D$ and $C - D$ are rule complexes.

Proof Note that $C \cap D, C - D \subseteq C$ and apply Definition 2.2. ■

Let us notice that rule complexes are well-founded.

Observation 2.6. There is no infinite sequence of rule complexes C_0, C_1, C_2, ... such that $\ldots \in C_2 \in C_1 \in C_0$. ■

[4] Finiteness of rule complexes is assumed for the sake of representation of social actors and interactions. From a theoretical point of view, rule complexes may be defined as infinite objects as well.

Theorem 2.7. C is a rule complex iff C is of one of the forms (1)–(4):

(1) $C = \emptyset$
(2) $(\exists x_0, \ldots, x_m)(rul(x_0) \wedge \ldots \wedge rul(x_m) \wedge C = \{x_0, \ldots, x_m\})$
(3) $(\exists x_0, \ldots, x_m)(compl(x_0) \wedge \ldots \wedge compl(x_m) \wedge C = \{x_0, \ldots, x_m\})$
(4) $(\exists x_0, \ldots, x_m)(\exists 0 \leq k < m)(rul(x_0) \wedge \ldots \wedge rul(x_k) \wedge$
$compl(x_{k+1}) \wedge \ldots \wedge compl(x_m) \wedge C = \{x_0, \ldots, x_m\})$

Proof ($\Leftarrow$) If C is of the form (1) or (2), then C is a rule complex by definition. Assume that C is of the form (3), i.e., $C = \{x_0, \ldots, x_m\}$ for some rule complexes $x_0, \ldots, x_m$. By definition $x_0 \cup \ldots \cup x_m$ is a rule complex, so is $\wp(x_0 \cup \ldots \cup x_m)$. Since $C \subseteq \wp(x_0 \cup \ldots \cup x_m)$, C is a rule complex by definition. Now assume that C is of the form (4), i.e., $C = \{x_0, \ldots, x_m\}$ for some rules $x_0, \ldots, x_k$ and rule complexes $x_{k+1}, \ldots, x_m$. Thus, C is a rule complex as a union of $\{x_0, \ldots, x_k\}$ and $\{x_{k+1}, \ldots, x_m\}$.
($\Rightarrow$) Let C be a rule complex. If C is a finite set of rules, then C is of the form (1) or (2). Notice that a union of two finite sets of rules (rule complexes) is a finite set of rules (rule complexes) and a union of a set of the form (4) and a set X of one of the forms (1)–(4) is of the form (4). In summary, if C is a union of two sets of the forms (1)–(4), then C is of one of the forms (1)–(4) as well. It is easy to see that if C is a subset of a set of one of the forms (1)–(4), then C is of one of the forms (1)–(4) as well. Finally, suppose that C is the power set of a set D of one of the forms (1)–(4). Thus, $C = \{X \mid X \subseteq D\}$. Subsets of D are rule complexes since they are of the forms (1)–(4). Hence C is of the form (3). ■

2.2.1. Rule part and complex part

We define the *complex part* of a rule complex C, $cp(C)$, as the set

$$cp(C) \stackrel{\text{def}}{=} \{x \in C \mid compl(x)\} \tag{2}$$

and the *rule part* of C, $rp(C)$, as the set

$$rp(C) \stackrel{\text{def}}{=} \{x \in C \mid rul(x)\}. \tag{3}$$

Proposition 2.8. For any rule complexes C, D, we have that:

$$rp(C) \cup cp(C) = C \quad \text{and} \quad rp(C) \cap cp(C) = \emptyset$$
$$rp(C \cup D) = rp(C) \cup rp(D) \quad \text{and} \quad cp(C \cup D) = cp(C) \cup cp(D)$$
$$rp(C \cap D) = rp(C) \cap rp(D) \quad \text{and} \quad cp(C \cap D) = cp(C) \cap cp(D)$$

$$rp(C - D) = rp(C) - rp(D) \quad \text{and} \quad cp(C - D) = cp(C) - cp(D)$$
$$rp(\wp(C)) = \emptyset \quad \text{and} \quad cp(\wp(C)) = \wp(C)$$

■

The proof is left as an exercise.

Example 2.9. Consider rule complexes from Example 2.3. Their rule and complex parts are as follows: $rp(C_1) = C_1$, $rp(C_2) = C_2$, $rp(C_3) = \{r_1\} = rp(C_4)$, $cp(C_1) = cp(C_2) = \emptyset$, $cp(C_3) = \{C_1\}$, and $cp(C_4) = \{C_2, C_3\}$. ■

2.2.2. Degree of a rule complex

With every rule complex C we can associate a natural number, $dg(C)$, indicating the degree of complexity of C. The *degree* of a rule complex C, $dg(C)$, is defined as follows:

$$dg(C) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } C = \emptyset \\ 1 & \text{if } C \neq \emptyset \text{ and } cp(C) = \emptyset \\ n+1 & \text{if } cp(C) \neq \emptyset \text{ and } \max\{dg(D) \mid D \in cp(C)\} = n \end{cases} \tag{4}$$

Example 2.10. Consider the rule complexes from Example 2.3. $dg(C_1) = 1 = dg(C_2)$, $dg(C_3) = 2$, and $dg(C_4) = 3$. ■

Proposition 2.11. For any rule complexes C, D, we have that:

(1) $dg(C) = 1$ iff $(cp(C) = \emptyset \wedge C \neq \emptyset)$ or $cp(C) = \{\emptyset\}$
(2) If $C \in D$, then $dg(C) + 1 \leq dg(D)$.
(3) If $C \subseteq D$, then $dg(C) \leq dg(D)$.
(4) $dg(C \cup D) = \max\{dg(C), dg(D)\}$
(5) $dg(C \cap D) \leq dg(C)$ and $dg(C - D) \leq dg(C)$
(6) $dg(\wp(C)) = dg(C) + 1$

Proof Let C, D be rule complexes. For (1) notice that if $C = \emptyset$, then $dg(C) = 0 \neq 1$ by definition. Now suppose that $cp(C) \neq \emptyset$ and $cp(C) \neq \{\emptyset\}$. Hence there is a non-empty rule complex D such that $D \in C$. Since $1 \leq dg(D)$, $2 \leq dg(C)$. The right-to-left part follows directly from the definition.
For (2) assume that $C \in D$. Hence $dg(C) \leq \max\{dg(C_0) \mid C_0 \in cp(D)\}$. Thus, $dg(C) + 1 \leq dg(D)$.
For (3) assume $C \subseteq D$. If $D = \emptyset$, then $C = \emptyset$ and we are done. If $cp(D) = \emptyset$ and $D \neq \emptyset$, then $dg(D) = 1$. In this case $cp(C) = \emptyset$ and hence $dg(C) \leq 1$. Suppose that $cp(D) \neq \emptyset$. Hence $1 \leq dg(D)$. If $cp(C) = \emptyset$, then $dg(C) \leq 1$ and we are done. Suppose that $cp(C) \neq \emptyset$. For each rule complex $C_0 \in C$, $dg(C_0) \leq \max\{dg(C_1) \mid C_1 \in cp(D)\}$. Hence $\max\{dg(C_0) \mid C_0 \in cp(C)\} \leq \max\{dg(C_0) \mid C_0 \in cp(D)\}$. Finally, $dg(C) \leq dg(D)$.

For (4) we have to consider a few cases. Let $C_0 = C \cup D$. Note that $C_0 = \emptyset$ iff both C and D are empty. Thus, $dg(C_0) = 0$ iff $dg(C) = 0$ and $dg(D) = 0$. Hence $dg(C_0) = 0 = \max\{dg(C), dg(D)\}$. Now assume that (a) $C_0 \neq \emptyset$ and (b) $cp(C_0) = \emptyset$. By (a), $C \neq \emptyset$ or $D \neq \emptyset$ and by (b), $cp(C) = \emptyset$ and $cp(D) = \emptyset$. Hence $dg(C_0) = 1 = \max\{dg(C), dg(D)\}$. Finally, consider the case that $cp(C_0) \neq \emptyset$. Consider any rule complex $C_1 \in C_0$. Hence $C_1 \in C$ or $C_1 \in D$. As a consequence, $dg(C_1) + 1 \leq dg(C)$ or $dg(C_1) + 1 \leq dg(D)$. Hence $dg(C_0) = \max\{dg(C_1) \mid C_1 \in cp(C_0)\} + 1 \leq \max\{dg(C), dg(D)\}$. On the other hand, $dg(C) \leq dg(C_0)$ and $dg(D) \leq dg(C_0)$ by (3). Hence $\max\{dg(C), dg(D)\} \leq dg(C_0)$.
Property (5) is an easy consequence of (3).
For (6) notice that $dg(\wp(C)) = \max\{dg(D) \mid D \subseteq C\} + 1$. For each $C_0 \subseteq C$, $dg(C_0) \leq dg(C)$ by (3). Hence $dg(\wp(C)) = dg(C) + 1$ as required. ■

2.2.3. Rule base and complex base

All rule complexes (resp., rules) constituting a rule complex form its complex (rule) base. Formally, a *complex base* of C, $cb(C)$, is the least set of rule complexes obtained according to the following formation rules:

$$cp(C) \subseteq cb(C)$$

$$(\forall x, y)((compl(x) \wedge x \in y \in cb(C)) \rightarrow x \in cb(C)) \qquad (5)$$

Now we define a *rule base* of C, $rb(C)$, as the following set:

$$rb(C) \stackrel{\text{def}}{=} rp(C \cup \bigcup cb(C)) \qquad (6)$$

The meaning of the two notions is explained by the following example.

Example 2.12. Let r_0, r_1, r_2, and r_3 be rules, $C = \{r_0, r_1, C_0, C_1\}$, $C_0 = \{r_2, r_3, C_2\}$, $C_1 = \{r_0, C_0\}$, and $C_2 = \{r_1\}$. $cb(C) = \{C_0, C_1, C_2\}$ and $rb(C) = \{r_0, r_1, r_2, r_3\}$. ■

Observation 2.13. For any rule complex C, $C \subseteq rb(C) \cup cb(C)$. ■

From the standpoint of social game theory, some rule complexes may have undesired properties. For instance, non-empty rule complexes with empty rule base seem to be useless. Let C be a rule complex. If $rb(C) = \emptyset$, the informative content of C is trivial. Therefore C is called *trivial.*

Example 2.14. Let r_0, r_1 be rules, $C = \{r_0, r_1, C_0, C_1, C_2\}$, $C_0 = \{r_1, C_3\}$, $C_3 = \{\emptyset\}$, $C_1 = \emptyset$, and $C_2 = \{C_1, C_3\}$. $cb(C) = \{C_0, C_1, C_2, C_3\}$. C_i is trivial for $i = 1, 2, 3$ since $rb(C_i) = \emptyset$. C_0 and C are not trivial since $rb(C_0) = \{r_1\}$ and $rb(C) = \{r_0, r_1\}$. ■

Also multiple occurrences of parentheses can be unimportant for representation of social actors and interactions.

Example 2.15. Let r_0, r_1 be rules, $C = \{r_0, C_0\}$, $C_0 = \{C_1\}$, and $C_1 = \{r_1\}$. That is, $C = \{r_0, \{\{r_1\}\}\}$. Intuitively, replacement of C by a simpler rule complex $\{r_0, \{r_1\}\}$ should not be harmful. ■

In our framework we can eliminate trivial rule complexes and redundant parentheses from a given rule complex whenever needed.

2.2.4. Generalised elements

Clearly, x is an element of a rule complex C, $x \in C$, iff $x \in rp(C) \cup cp(C)$. We generalise the notion of membership as follows. x is a *generalised element* (or *g-element*) of C, $x \in_g C$, in case x is an element of the rule or complex base of C.

$$x \in_g C \stackrel{\text{def}}{\leftrightarrow} x \in rb(C) \cup cb(C) \qquad (7)$$

Two different rule complexes may have the same g-elements.

Example 2.16. Consider C from Example 2.12 and $C_3 = \{C_0, C_1\}$. Rule complexes C and C_3 are different but have the same g-elements. ■

Proposition 2.17. For any rule or rule complex x and rule complexes C, D, we have that:

If $x \in C$, then $x \in_g C$.
If $x \in_g C \in_g D$, then $x \in_g D$.

■

The easy proof is left as an exercise. Now consider a set of rules $\mathcal{R}$ and a family of rule complexes $\mathcal{C}$ formed of elements of $\mathcal{R}$.

Corollary 2.18. The relation $\in_g \subseteq (\mathcal{R} \cup \mathcal{C}) \times \mathcal{C}$ is a transitive closure of $\in$. ■

2.2.5. Subcomplexes

The notion of subcomplex of a given rule complex, presented here in informal way only, generalises the notion of subset and is of great importance for representation of social actors and their interactions. Observe that a set X is a subset of a set Y if $X = Y$ or X obtains from Y by deletion of some (or all) elements of Y. Intuitively, a rule complex C is a *subcomplex* of a rule complex D, $C \subseteq_g D$, if $C = D$ or C obtains from D by deletion of some (or all) occurrences of g-elements of D or removal of redundant parentheses.

Example 2.19. Consider the rule complex C from Example 2.12. Let $C_3 = \{r_2, C_2\}$, $C_4 = \{r_0, C_5\}$, and $C_5 = \{r_2, r_3\}$. $C_6 = \{r_1, C_3, C_4\}$ is a subcomplex of C. Indeed, C_6 is obtained from C by deleting of the occurrence of r_0 in C, the occurrence of r_3 in the occurrence of C_0 in C, and the occurrence of C_2 in the occurrence of C_0 in the occurrence of C_1 in C. ■

The following properties can be proved about subcomplexes. The proof is easy but requires a formal definition of the notion of subcomplex, and hence it is omitted here.

Proposition 2.20. For any rule complexes C_0, C, and D, we have that:

If $C \subseteq D$, then $C \subseteq_g D$.

$C \subseteq_g C$.

If $C \in_g D$, then $C \subseteq_g D$.

If $C_0 \subseteq_g C \wedge C \subseteq_g D$, then $C_0 \subseteq_g D$.

■

3. Derivability

We introduce a formal notion of derivability of a rule from a rule complex[5]. Our notion of derivability is closely related to the notion of extension of a default theory [4][6]. Henceforth the set of all rules derived from a rule complex C will be denoted by *Der*(C). We also define a formal notion of consistency of a rule complex, viz., *Der*-consistency. Recall that $r(\alpha) = \{(\emptyset, \emptyset, \alpha)\}$.

Definition 3.1. The set of all rules *derived* from a finite set of rules C, *Der*(C), is defined as follows.

(1) Let r be an axiomatic rule.

(i) $r \in Der^0(C) \stackrel{\text{def}}{\leftrightarrow} r \in C$

(ii) $r \in Der^{k+1}(C) \stackrel{\text{def}}{\leftrightarrow} (\forall\gamma)((\emptyset, \emptyset, \gamma) \in r \rightarrow (\exists r' \in C)(\exists X, Y)((X, Y, \gamma) \in r' \wedge (\forall\alpha \in X) r(\alpha) \in Der^k(C) \wedge (\forall\beta \in Y) r(\neg\beta) \notin Der(C)))$

(iii) $Der(C) \stackrel{\text{def}}{=} \bigcup_{k \in \mathcal{N}} Der^k(C)$

[5] The notion defined in the present paper is a simple version of derivability. For simplicity, we forget about the structure of rule complexes and represent rule complexes by their rule bases which are finite sets of rules.

[6] Notice the occurrence of *Der*(C) in the definition of $Der^{k+1}(C)$.

(2) Let r be any rule.

$r \in Der(C) \stackrel{\text{def}}{\leftrightarrow} (\forall X, Y)(\forall\gamma)(((X, Y, \gamma) \in r \wedge (\forall\beta \in Y) r(\neg\beta) \notin Der(C \cup \{r(\alpha) \mid \alpha \in X\})) \rightarrow r(\gamma) \in Der(C \cup \{r(\alpha) \mid \alpha \in X\}))$

Given any rule complex C, the set of all rules *derived* from C, *Der*(C), is defined as follows:

$$Der(C) \stackrel{\text{def}}{=} Der(rb(C)) \tag{8}$$

Proposition 3.2. Let C be any rule complex, r a rule, and α a formula.

If $r \in_g C$, then $r \in Der(C)$.

If α is not a conclusion of any $r \in_g C$, then $r(\alpha) \notin Der(C)$.

■

Let

$$active(r, C) = \{(X, Y, \gamma) \in r \mid (\forall\alpha \in X) r(\alpha) \in Der(C) \wedge \wedge (\forall\beta \in Y) r(\neg\beta) \notin Der(C)\}. \tag{9}$$

A rule r is *active* w.r.t. a rule complex C if $active(r, C) \neq \emptyset$.

The notion of consistency (in the traditional sense) of a set of formulae may be generalised to the case of rule complexes. A rule complex C is *Der-consistent* if there is no formula γ such that $r(\gamma), r(\neg\gamma) \in Der(C)$; otherwise it is *Der-inconsistent.* By Proposition 3.2, if there are no rules $r_0, r_1 \in_g C$ such that γ is the conclusion of r_0 and $\neg\gamma$ is the conclusion of r_1, for some formula γ, then C is *Der*-consistent.

Example 3.3. For simplicity we consider the schematic rules case. Assume that all formulae α_i are different and $\alpha_i \neq \neg\alpha_j$ for $i, j = 0, \ldots, 8$. Consider the following rules: $r_0 : \frac{}{\alpha_0}$, $r_1 : \frac{}{\alpha_1}$, $r_2 : \frac{:\alpha_2}{\neg\alpha_3}$, $r_3 : \frac{\alpha_1 : \alpha_4}{\alpha_3}$, $r_4 : \frac{\alpha_0 :}{\neg\alpha_2}$, $r_5 : \frac{\neg\alpha_1 : \alpha_4}{\neg\alpha_5}$, $r_6 : \frac{\alpha_0, \alpha_6, \alpha_7 :}{\neg\alpha_8}$, $r_7 : \frac{\alpha_0, \alpha_3 : \alpha_5}{\alpha_8}$, and $r_8 : \frac{\alpha_6, \alpha_7 : \alpha_5}{\neg\alpha_8}$. Let $D = \{r_0, \ldots, r_6\}$ and $D' = D - \{r_4\}$. The rule r_7 is active w.r.t. but not derivable from D. On the other hand, the rule r_8 is derived from but not active w.r.t. D. The same can be proved for any rule complex C such that $rb(C) = D$. The rule complex D' is *Der*-inconsistent since we can derive the rule $\frac{}{\alpha_3}$ by r_1, r_3 and the rule $\frac{}{\neg\alpha_3}$ by r_2. In the case of D, the derivation of $\frac{}{\neg\alpha_3}$ is blocked. By r_0 and r_4 we derive $\frac{}{\neg\alpha_2}$. Then we cannot apply r_2. In summary, D is *Der*-consistent. ■

4. Application of rules and rule complexes

In this section we briefly discuss the question of application of rules and rule complexes in a given situation. The problem is of great importance since actors act and interact by application of various game orders and action complexes, attempt to realise their goals and values by application of appropriate rule complexes, and make judgments by application of suitable or available judgment complexes. Application of rules involves interpretation, situational understanding, and considerable improvisation. Specification of rules is usually incomplete and, hence, rules have to be interpreted and applied using situational information and knowledge. Application of rule complexes involves similar components as application of single rules and, additionally, judgment making which rules of a given rule complex to apply and in which order. Application of rules and rule complexes is context-dependent, viz., rules and rule complexes applied in one situation or by one actor may be not applicable in another situation or by another actor. Moreover, social actors often adapt rules or innovate in varying situations.

For simplicity assume that a considered actor a_i has unbounded reasoning capabilities. Situations can be partially described by actors' models of these situations and, hence, can be represented by rule complexes. Thus, a given situation S_t may be represented by a rule complex $MOD(i,t)$ consisting of a_i's explicit beliefs about S_t. A fact represented by α (or simply, formula α) *holds* in S_t in a_i's opinion if the rule $r(\alpha)$ is derivable from a_i's model of S_t at t, written $r(\alpha) \in Der(MOD(i,t))$. On the other hand, α *may hold* in S_t in a_i's opinion if $\neg\alpha$ does not hold in S_t up to a_i's mind, i.e., if $r(\neg\alpha) \notin Der(MOD(i,t))$. A rule r is *active* in S_t in a_i's opinion if r is active w.r.t. $MOD(i,t)$. If a rule is active in a given situation from an actor's perspective, then the actor can try to apply it.

Unless r orders to perform or not an action, then we define the set of consequences of application of r by a_i at t, $Con(r,i,t)$, as the set $Con(r,i,t) \stackrel{\text{def}}{=} \{r'\}$ where

$$r' = \{(\emptyset,\emptyset,\gamma) \mid (\exists X,Y)(X,Y,\gamma) \in active(r, MOD(i,t))\}. \tag{10}$$

If r orders not to perform some action(s), then the set of consequences is defined as the empty set, i.e., $Con(r,i,t) = \emptyset$. Suppose that r orders to do some action(s). Usually only a finite (even if very large) number of results of executed actions are of interest to an actor. Let formulae $\gamma_0,\ldots,\gamma_k$ represent these results. We define the set of consequences as the set $Con(r,i,t) = \{r(\gamma_0),\ldots,r(\gamma_k)\}$.

In practice, application of a rule r by a_i will transform S_t into a (possibly new) situation S_{t+1}. In case a_i tries to predict what a situation will obtain if r is applied at t, a_i will usually get a class of possible situations. For simplicity let us consider the first case. Recall that situations are represented by a_i's models of these situations. Given an a_i's model $MOD(i,t)$ of S_t, the set of consequences $Con(r,i,t)$ of application of r by a_i at t, and an updating mapping[7] $\otimes$, we define the *result* of application of r by a_i at t, $MOD(i,t+1)$, as the rule complex

$$MOD(i,t+1) \stackrel{\text{def}}{=} MOD(i,t) \otimes Con(r,i,t). \tag{11}$$

In words, the result of application of r by a_i at t is defined as the result of updating of a_i's model of S_t by consequences of application of r by a_i at t.

It is difficult to say how to apply a rule complex in a given situation in general. Depending on a_i's actual modality of actioning, a_i may orient him or herself on realisation of some goal(s), on the application as such, etc. In all these cases the way of application may be different. However, there are some elements in common. Suppose C_0 is a rule complex to be applied by a_i in S_t. If no rule of C_0 is active in S_t, C_0 cannot be applied. Thus, suppose that X_0 is a non-empty set of rules of C_0 active in S_t. Collective social actors usually apply rules concurrently. Human actors mostly apply rules sequentially but they can also apply rules concurrently in some cases. For simplicity assume that a_i applies rules sequentially. In the next step a_i has to decide which rule of X_0 to apply. Suppose r_0 is such a rule. If a_i succeeds, then $MOD(i,t+1)$ described by (11), where r is replaced by r_0, is taken as the representation of S_{t+1}. In this situation a_i has to apply a rule complex C_1 being C_0 or possibly another rule complex. Let X_1 be the set of rules of C_1 active in S_{t+1} in a_i's opinion. Again, a_i chooses a rule r_1 to be applied, tries to apply it, etc. The process of application terminates[8] at some point $t+m$. Thus, we obtain a finite sequence of models $MOD(i,t), MOD(i,t+1),\ldots,MOD(i,t+m)$ such that for any $j = 0,\ldots,m-1$,

$$MOD(i,t+j+1) = MOD(i,t+j) \otimes Con(r_j,i,t+j). \tag{12}$$

Finally, we say that the result of application of C_0 by a_i at t is a situation S_{t+m}, represented by $MOD(i,t+m)$ as above.

[7] Transformations of rule complexes, and updating among others, will be discussed in a separate paper.

[8] Theoretically, it would continue interminably, e.g., if a_i tried to apply mutual meta-rules ordering to apply each other. In practice, social actors can stop the process of application of a rule complex if necessary or convenient.

5. Summary

In the paper the notion of rule complex, several associated notions, and their elementary properties were presented. Rule complexes seem to be a promising tool to represent both social actors and their interactions. In spite of their finiteness, rule complexes may have a complicated nature. This gives rise to asking many interesting questions about their properties. For lack of space we only touched upon such problems as subcomplexes, derivability, (in)consistency or application of rule complexes. These and other questions (e.g., transformations) will be discussed more exhaustively later on.

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TENSE LOGIC WITHOUT THE PRINCIPLE OF THE EXCLUDED MIDDLE

Summary

In the article [2] was proposed intuitionistic tense logic system based on unchanging time. In this paper we construct intuitionistic tense logic system based on changing time. The system proposed in [2] is a particular case of the system presented in this work.

According to a common view, to create an indeterministic tense logic system we have to reject either the principle of causality or the principle of bivalency or the excluded middle law. However, Łukasiewicz in paper "On determinism" showed, that the rejection of the principle of causality is not sufficient: to create an indeterministic tense logic system it is necessary to reject either the principle of bivalency or the excluded middle law. It seems that a certain composition of the tense propositional logic and the intuitionistic propositional logic is the proper tool for describing future events from indeterministic viewpoint. The constructed system describes time which posses a changing structure. Let us consider a set of states of knowledge. The set is ordered by the relation: not lesser. A state of knowledge may be enlarged to a not lesser one in several cases. One of them is when in the new state of knowledge we are able to describe events which were not known in the previous state. Another case is when we obtain a new knowledge about the structure of time. For example: about the branching. One more case is when in a new state of knowledge the relation between time points is changed. In the proposed semantics we consider an indexed set of states of knowledge, where: m_i is the state of knowledge with the index i, T_i is the set of time points known in a state m_i, R_i is a binary relation on a set T_i. For every state of knowledge m_i we have a particular relation. This relation is understood as a relation "earlier-later" in a state of knowledge m_i. V_i is

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a function mapping the set T_i of time points to the set of subsets of propositional letters. Every state of knowledge describes a fragment of the structure of time which is known in a given state of knowledge. Time in a state of knowledge m_i is an ordered pair $\langle T_i, R_i\rangle$. In the considered system time is understood as the sum of times from all states of knowledge.

The system IT_m

The system IT_m is the tense logic system based on intuitionistic propositional logic. The tense-logical propositional language $\mathfrak{L}$ consists of:

1) $p_1, p_2, p_3, \ldots$ – propositional letters
2) $\neg$ – intuitionistic unary connectives
3) $\wedge, \vee, \rightarrow$ – intuitionistic binary connectives
4) F, G, P, H – tense operators
5)), (– parentheses.

The tense operators are defined in the usual way:

F – "at least once in the future"
G – "it is always going to be the case"
P – "at least once in the past"
H – "it has always been the case"

Let I be a nonempty set of indexes of states of knowledge. Let T_i $(i \in I)$ be a nonempty set of time points known in a state knowledge indexed i.

Let R_i $(\subseteq T_i \times T_i)$ be an "earlier-later" relation on set T_i.

Set of ordered pairs $\{\langle T_i, R_i\rangle : i \in I\}$ is a set of times in particular states of knowledge.

Time $\mathcal{T}$ is understood as an ordered pair $\left\langle \bigcup_{i\in I} T_i, \bigcup_{i\in I} R_i \right\rangle$, where the set of time points is understood as the sum of sets of time points from particular states of knowledge and the relation "earlier-later" is understood as the sum of relations "earlier-later" from particular states of knowledge $\left(\bigcup_{i\in I} R_i \subseteq \bigcup_{i\in I} T_i \times \bigcup_{i\in I} T_i \right)$.

Let V_i be a function mapping elements $t \in T_i$ to subsets $V_i(t)$ of the set of propositional letters. Let $\mathcal{F}$ be a nonempty class of such functions $\mathcal{F} = \{V_i\}_{i\in I}$.

A state of knowledge m_i is an ordered triple $m_i = \langle T_i, R_i, V_i\rangle$

$\mathfrak{M}_{(\mathcal{T},\mathcal{F})}$ is a model based on time $\mathcal{T}$ and the class of functions $\mathcal{F}$

$\mathfrak{M}_{(\mathcal{T},\mathcal{F})} = \{\langle T_i, R_i, V_i\rangle : V_i \in \mathcal{F}, i \in I\}$

Between elements of a model $\mathfrak{M}_{(\mathcal{T},\mathcal{F})}$: $m_i = \langle T_i, R_i, V_i\rangle$, $m_j = \langle T_j, R_j, V_j\rangle$ we introduce relation $\leq$ defined as follows:

Definition 1

$$m_i \leq m_j \stackrel{\text{def}}{\equiv} (T_i \subseteq T_j \quad \wedge \quad R_i \subseteq R_j \quad \wedge \quad \forall t \in T_i \;\; V_i(t) \subseteq V_j(t)).$$

$m_i \leq m_j$ means that the state m_j is not lesser than the state m_i. Defined relation is reflexive and transitive.

T1

For any m_i holds $m_i \leq m_i$

T2

For any m_i, m_j, m_k holds: if $(m_i \leq m_j$ and $m_j \leq m_k)$ then $m_i \leq m_k$

The facts T1 and T2 we can easily prove using basic properties of the inclusion relation.

Remark

m_i^* (where $m_i^* = \langle T_i^*, R_i^*, V_i^*\rangle$) means any m_j such that $m_i \leq m_j$.

For a model $\mathfrak{M}_{(\mathcal{T},\mathcal{F})}$, a state of knowledge $m_i = \langle T_i, R_i, V_i\rangle$, an element $t \in T_i$, a tense-logical formula α, "truth definition" in model $\mathfrak{M}_{(\mathcal{T},\mathcal{F})}$ is defined the following conditions:

Definition 2

$\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \alpha[t, m_i]$ is defined the following conditions:

a) $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models p[t, m_i] \stackrel{\text{def}}{\equiv} p \in V_i(t)$.

b) $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \neg\alpha[t, m_i] \stackrel{\text{def}}{\equiv} \forall m_i^* \in \mathfrak{M}_{(\mathcal{T},\mathcal{F})} \;\; \mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models \alpha[t, m_i^*]$

c) $\mathfrak{M}_{\mathcal{T},\mathcal{F})} \models (\alpha \vee \beta)[t, m_i] \stackrel{\text{def}}{\equiv} \mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \alpha[t, m_i]$ or $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \beta[t, m_i]$

d) $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models (\alpha \wedge \beta)[t, m_i] \stackrel{\text{def}}{\equiv} \mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \alpha[t, m_i]$ and $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \beta[t, m_i]$

e) $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models (\alpha \rightarrow \beta)[t, m_i] \stackrel{\text{def}}{\equiv} \forall m_i^* \in \mathfrak{M}_{(\mathcal{T},\mathcal{F})} \;\; \mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models \alpha[t, m_i^*]$ or $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \beta[t, m_i^*])$

f) $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models F\alpha[t, m_i] \stackrel{\text{def}}{\equiv} \exists t_1 \in T_i \;\; tR_it_1$ such that $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \alpha[t_1, m_i]$

g) $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models G\alpha[t, m_i] \stackrel{\text{def}}{\equiv} \forall m_i^* \; \forall t_1 \in T_i^* \;\; (tR_i^*t_1$ implies $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \alpha[t_1, m_i^*])$

8) $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models P\alpha[t, m_i] \stackrel{\text{def}}{\equiv} \exists t_1 \in T_i \;\; t_1R_it$ such that $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \alpha[t_1, m_i]$

h) $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models H\alpha[t, m_i] \quad \stackrel{\text{def}}{\equiv} \quad \forall m_i^* \, \forall t_1 \in T_i^* \;$ ($t_1 R_i^* t$ implies $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \alpha[t_1, m_i^*]$)

Let us remark that if for some state of knowledge m_i and for some element $t \in T_i$ the formula $F\alpha$ is true, then in this state of knowledge there is time point latter than element t such that the formula α is true in this time. For making a decision that the formula $F\alpha$ is true in the some state of knowledge m_i suffices an analysis of structure of time only given state of knowledge. From the definition of the relation $\leq$ we have that: if a formula $F\alpha$ is true in a state of knowledge m_i then it is true in any state of knowledge not lesser than the state m_i.

In case of a formula $G\alpha$ we have another situation. According to interpretation of G operator the formula $G\alpha$ is understood as "always in the future will be that α". Hence the truth of the formula $G\alpha$ in a state m_i and time $t \in T_i$ implies the truth of the formula α in any state of knowledge not lesser than m_i and in any element t_1 later than t. We can not consider the truth of the formula $G\alpha$ only in limits of a given state of knowledge. Statement, that a formula α will be true always in the future obligates us to assure the truth of this formula in any element of the future time – even outside the actual state of knowledge.

The truth of a formula α in a model based on time $\mathcal{T}$ and a class of functions $\mathcal{F}$ means that the formula α is true in any time point of any state of knowledge in this model.

Definition 3

$\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \alpha$ (α is true in a model $\mathfrak{M}_{(\mathcal{T},\mathcal{F})}$) iff for any $m_i \in \mathfrak{M}_{(\mathcal{T},\mathcal{F})}$ and for any $t \in T_i$ holds $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \alpha[t, m_i]$.

Definition 4

$\mathcal{T} \models \alpha$ (α is true in time $\mathcal{T}$) iff is true in a model $\mathfrak{M}_{(\mathcal{T},\mathcal{F})}$ for any nonempty class of functions $\mathcal{F}$.

Definition 5

$\models \alpha$ (α is a tautology of intuitionistic tense logic) iff is true in any time $\mathcal{T}$.

In empirical sciences we often have a situation, that in certain state of knowledge we accept some theses as true and in a not lesser state of knowledge these theses are verified. However, in considered system the whole knowledge from certain state of knowledge is admitted in the not lesser state of knowledge. For the considered system an adequate model of knowledge

is for example mathematical knowledge. A theorem once correctly proved is accepted. Advancement of mathematics does not produce the rejection of the theorem proved earlier. The property of remaining of knowledge in a not lesser state of knowledge is expressed by the following lemma:

Lemma

For any formula α and for any m_i, m_j such that $m_i \leq m_j$ holds:

$$\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \alpha[t, m_i] \text{ implies } \mathfrak{M}_{(\mathcal{T},\mathcal{F})} \models \alpha[t, m_j]$$

IT_m contains an axiom system of intuitionistic propositional logic (A1-A10), tense logical axioms (H1-GD'), rules *Modus Ponens* (MP) and the tense-logical rules (RH and RG)

Axioms

For any $\alpha, \beta, \gamma \in \mathcal{L}$:

Intuitionistic propositional logic axioms:

A1) $\alpha \to (\beta \to \alpha)$

A2) $(\alpha \to \beta) \to \{[\alpha \to (\beta \to \gamma)] \to (\alpha \to \gamma)\}$

A3) $[(\alpha \to \gamma) \wedge (\beta \to \gamma)] \to \{(\alpha \vee \beta) \to \gamma\}$

A4) $(\alpha \wedge \beta) \to \alpha$

A5) $(\alpha \wedge \beta) \to \beta$

A6) $\alpha \to [\beta \to (\alpha \wedge \beta)]$

A7) $\alpha \to (\alpha \vee \beta)$

A8) $\beta \to (\alpha \vee \beta)$

A9) $(\alpha \wedge \neg\alpha) \to \beta$

A10) $(\alpha \to \neg\alpha) \to \neg\alpha$

Tense-logical axioms:

H1) $H(\alpha \Rightarrow \beta) \Rightarrow (H\alpha \Rightarrow H\beta)$

H2) $H(\alpha \Rightarrow \beta) \Rightarrow (P\alpha \Rightarrow P\beta)$

H3) $\alpha \Rightarrow HF\alpha$

H4) $PG\alpha \Rightarrow \alpha$

H5) $P(\alpha \vee \beta) \Rightarrow (P\alpha \vee P\beta)$

H6) $(P\alpha \Rightarrow H\beta) \Rightarrow H(\alpha \Rightarrow \beta)$

H7) $P(\alpha \Rightarrow \beta) \Rightarrow (H\alpha \Rightarrow P\beta)$

H8) $P\alpha \Rightarrow \neg H \neg\alpha$

G1) $G(\alpha \Rightarrow \beta) \Rightarrow (G\alpha \Rightarrow G\beta)$

G2) $G(\alpha \Rightarrow \beta) \Rightarrow (F\alpha \Rightarrow F\beta)$

G3) $\alpha \Rightarrow GP\alpha$

G4') $FH\alpha \Rightarrow \alpha$

G5) $F(\alpha \vee \beta) \Rightarrow (F\alpha \vee F\beta)$

G6) $(F\alpha \Rightarrow G\beta) \Rightarrow G(\alpha \Rightarrow \beta)$

G7) $F(\alpha \Rightarrow \beta) \Rightarrow (G\alpha \Rightarrow F\beta)$

G8) $F\alpha \Rightarrow \neg G \neg\alpha$

Rules:

MP: $\dfrac{\alpha \to \beta, \alpha}{\beta}$

RH: $\dfrac{\alpha}{H\alpha}$

RG: $\dfrac{\alpha}{G\alpha}$

The axioms H1, G1, H2, G2 are axioms in K_t system. The other axioms are theorems of K_t system. In T_m system we can not prove these theorems because intuitionistic logic does not accept some nonconstructive laws of the classical logic. In tense logic systems based on classical logic the operators G, F and H, P are dual. $P : P\alpha \leftrightarrow \neg H \neg \alpha$ (analogous for the operators G and F). However, in the considered system we have not got such duality. For the sake of obeing laws and rules of intuitionistic logic the reverses of the implications expressed in axioms H8 and G8 does not hold.

The system IT_m is a minimal system of the intuitionistic tense logic. No conditions are imposed upon R relation.

Some tense formulas which are deterministic in certain time structures are not tautologies of the IT_m. The formula $\alpha \Rightarrow HF\alpha$ is an axiom of the IT_m but it does not express determinism in IT_m. This formula is deterministic in a time structure with linear time order. In case branching time structure and our understood tense operators the formula $\alpha \Rightarrow HF\alpha$ is not deterministic.

The philosophical idea that the principle of the excluded middle is not applicable to the future events is fulfilled in the considered system because the formula $F\alpha \vee F\neg\alpha$ is not a tautology of the IT_m.

T3

The formula $F\alpha \vee F\neg\alpha$ is not a tautology of the IT_m.

Proof

Let us take the state of knowledge:

$m_1 = \langle T_1, R_1, V_1 \rangle$,

$m_2 = \langle T_2, R_2, V_2 \rangle$,

where $T_1 = T_2 = \{t_1, t_2\}$, $R_1 = R_2 = \{(t_1, t_2)\}$.

Let us take the functions V_1, V_2 such that:

1) $p \notin V_1(t_2)$.

2) $p \in V_2(t_2)$.

Let us take:

$\mathcal{T} = \langle T, R_2 \rangle$,

$\mathcal{F} = \{V_1, V_2\}$ and

$\mathfrak{M}_{(\mathcal{T},\mathcal{F})} = \{m_1, m_2\}$.

The state of knowledge m_2 will be not lesser than state of knowledge m_1. This means that:

$$m_1 \leq m_2.$$

We have:

$$\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models Fp\,[t_1, m_1].$$

Because hold $\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models Fp\,[t_1, m_2]$ then we do not have that:

$$\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models F\neg p\,[t_1, m_1].$$

In the other case we have:

$$\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models \neg p\,[t_2, m_1].$$

Because the state of knowledge m_2 is not lesser than the state of knowledge m_1, then whole knowledge from the state of knowledge m_1 is admited in the state of knowledge m_2. Hence we have that:

$$\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models \neg p\,[t_2, m_2].$$

It is a contradiction with 2).

We conclude that:

$$\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models F\neg p\,[t_1, m_1].$$

In the state of knowledge m_1 and the time t_1 we have:

$$\mathfrak{M}_{(\mathcal{T},\mathcal{F})} \not\models (Fp \vee F\neg p)\,[t_1, m_1].$$

Let us remark that in a proof that the formula $F\alpha \vee F\neg\alpha$ is not a tautology of the IT_m we do not use an assumption that a time possess the last element. In case structures wchich possess the last moment, the formula $F\alpha \vee F\neg\alpha$ is not a tautology because it is not true in the last element of time. In the last moment of time does not exist a later time element in which holds either $F\alpha$ either $F\neg\alpha$.

The formula $F\alpha \vee F\neg\alpha$ is not a tautology of the IT_m even in structures, which does not possess the last moment of time.

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SOME TECHNICAL RESULTS IN A CERTAIN INTUITIONISTIC TENSE LOGIC

In article [1] Ewald constructs intuitionistic tense logic system and calls it IK_t[1]. This system was commented in work [2]. Ewald adds the temporal connectives G, H, F and P to the connectives of propositional ituitionistic calculus and offers a Hilbert axiom system for the logic IK_t. In this paper we would like to show that the axioms proposed by Ewald form dependent axiom system on the basis of intuitionistic logic.

The axioms proposed by Ewald are the following:

A1) All axioms of the intuitionistic propositional calculus

A2) $G(\alpha \Rightarrow \beta) \Rightarrow (G\alpha \Rightarrow G\beta)$	A2') $H(\alpha \Rightarrow \beta) \Rightarrow (H\alpha \Rightarrow H\beta)$
A3) $G(\alpha \wedge \beta) \Leftrightarrow (G\alpha \wedge G\beta)$	A3') $H(\alpha \wedge \beta) \Leftrightarrow (H\alpha \wedge H\beta)$
A4) $F(\alpha \vee \beta) \Leftrightarrow (F\alpha \wedge F\beta)$	A4') $F(\alpha \vee \beta) \Leftrightarrow (F\alpha \wedge F\beta)$
A5) $G(\alpha \Rightarrow \beta) \Rightarrow (F\alpha \Rightarrow F\beta)$	A5') $H(\alpha \Rightarrow \beta) \Rightarrow (P\alpha \Rightarrow P\beta)$
A6) $(G\alpha \wedge F\beta) \Rightarrow F(\alpha \wedge \beta)$	A6') $(H\alpha \wedge P\beta) \Rightarrow P(\alpha \wedge \beta)$
A7) $G\neg\alpha \Rightarrow \neg F\alpha$	A7') $H\neg\alpha \Rightarrow \neg P\alpha$
A8) $FH\alpha \Rightarrow \alpha$	A8') $PG\alpha \Rightarrow \alpha$
A9) $\alpha \Rightarrow GP\alpha$	A9') $\alpha \Rightarrow HF\alpha$
A10) $(F\alpha \Rightarrow G\beta) \Rightarrow G(\alpha \Rightarrow \beta)$	A10') $(P\alpha \Rightarrow H\beta) \Rightarrow H(\alpha \Rightarrow \beta)$
A11) $F(\alpha \Rightarrow \beta) \Rightarrow (G\alpha \Rightarrow F\beta)$	A11') $P(\alpha \Rightarrow \beta) \Rightarrow (H\alpha \Rightarrow P\beta)$

Rules of inference:

Modus Ponens

$$MP: \quad \frac{\alpha \Rightarrow \beta, \alpha}{\beta}$$

[1] It is intuitionistic analogue to Lemmon's system K_t

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Temporal generalization rules:

RH: $\dfrac{\alpha}{H\alpha}$

RG: $\dfrac{\alpha}{G\alpha}$

We show that the axioms A3 and A3' are derivable from some of tautologies of intuitionistic logic and axioms respectively A2 and A2'.

T1: $G(\alpha \wedge \beta) \Leftrightarrow (G\alpha \wedge G\beta)$

proof

a) $G(\alpha \wedge \beta) \Rightarrow (G\alpha \wedge G\beta)$

1) $(\alpha \wedge \beta) \Rightarrow \alpha$ – tautology[2]
2) $(\alpha \wedge \beta) \Rightarrow \beta$ – tautology[3]
3) $G[(\alpha \wedge \beta) \Rightarrow \alpha]$ – 1, RG
4) $G[(\alpha \wedge \beta) \Rightarrow \beta]$ – 2, RG
5) $G[(\alpha \wedge \beta) \Rightarrow \alpha] \Rightarrow [G(\alpha \wedge \beta) \Rightarrow G\alpha]$ – A2
6) $G[(\alpha \wedge \beta) \Rightarrow \beta] \Rightarrow [G(\alpha \wedge \beta) \Rightarrow G\beta]$ – A2
7) $G(\alpha \wedge \beta) \Rightarrow G\alpha$ – 3, 5, MP
8) $G(\alpha \wedge \beta) \Rightarrow G\beta$ – 4, 6, MP
9) $[G(\alpha \wedge \beta) \Rightarrow G\alpha] \Rightarrow \{[G(\alpha \wedge \beta) \Rightarrow G\beta] \Rightarrow [G(\alpha \wedge \beta) \Rightarrow (G\alpha \wedge G\beta)]\}$ – tautology[4]
10) $[G(\alpha \wedge \beta) \Rightarrow G\beta] \Rightarrow [G(\alpha \wedge \beta) \Rightarrow (G\alpha \wedge G\beta)]$ – 7, 9, MP
11) $G(\alpha \wedge \beta) \Rightarrow (G\alpha \wedge G\beta)$ – 8, 10, MP

b) $(G\alpha \wedge G\beta) \Rightarrow G(\alpha \wedge \beta)$

1) $\alpha \Rightarrow [\beta \Rightarrow (\alpha \wedge \beta)]$ – tautology[5]
2) $G\{\alpha \Rightarrow [\beta \Rightarrow (\alpha \wedge \beta)]\}$ – 1, RG
3) $G\{\alpha \Rightarrow [\beta \Rightarrow (\alpha \wedge \beta)]\} \Rightarrow \{G\alpha \Rightarrow G[\beta \Rightarrow (\alpha \wedge \beta)]\}$ – A2
4) $G\alpha \Rightarrow G[\beta \Rightarrow (\alpha \wedge \beta)]$ – 2, 3, MP
5) $G[\beta \Rightarrow (\alpha \wedge \beta)] \Rightarrow [G\beta \Rightarrow G(\alpha \wedge \beta)]$ – A2
6) $G\alpha \Rightarrow [G\beta \Rightarrow G(\alpha \wedge \beta)]$ – 4, 5, hip. syll[6], MP
7) $\{G\alpha \Rightarrow [G\beta \Rightarrow G(\alpha \wedge \beta)]\} \Rightarrow [(G\alpha \wedge G\beta) \Rightarrow G(\alpha \wedge \beta)]$ – tautology[7]
8) $(G\alpha \wedge G\beta) \Rightarrow G(\alpha \wedge \beta)$ – 6, 7, MP

The thesis is conclusion from a) and b). □

T1': $H(\alpha \wedge \beta) \Leftrightarrow (H\alpha \wedge H\beta)$

Proof (Analogous to T1)

The following implication is derivable from the axiom A5 and some of tautologies of the intuitionistic logic.

T2: $(F\alpha \vee F\beta) \Rightarrow F(\alpha \vee \beta)$

Proof

1) $\alpha \Rightarrow (\alpha \vee \beta)$ – tautology[8]
2) $\beta \Rightarrow (\alpha \vee \beta)$ – tautology[9]
3) $G[\alpha \Rightarrow (\alpha \vee \beta)]$ – RG, 1
4) $G[\beta \Rightarrow (\alpha \vee \beta)]$ – RG, 2
5) $G[\alpha \Rightarrow (\alpha \vee \beta)] \Rightarrow [F\alpha \Rightarrow F(\alpha \vee \beta)]$ – A5
6) $G[\beta \Rightarrow (\alpha \vee \beta)] \Rightarrow [F\beta \Rightarrow F(\alpha \vee \beta)]$ – A5
7) $F\alpha \Rightarrow F(\alpha \vee \beta)$ – 3, 5, MP
8) $F\beta \Rightarrow F(\alpha \vee \beta)$ – 4, 6, MP

[2] The formula $(\alpha \wedge \beta) \Rightarrow \alpha$ is a tautology of the intuitionistic propositional calculus (see Pogorzelski, 1992, p. 41).

[3] The formula $(\alpha \wedge \beta) \Rightarrow \beta$ is a tautology of the intuitionistic propositional calculus (see Pogorzelski, 1992, p. 41).

[4] The formula $(\alpha \Rightarrow \beta) \Rightarrow \{(\alpha \Rightarrow \gamma) \Rightarrow [\alpha \Rightarrow (\beta \wedge \gamma)]\}$ is a tautology of the intuitionistic propositional calculus (see Grzegorczyk, p. 106).

[5] The formula $\alpha \Rightarrow [\beta \Rightarrow (\alpha \wedge \beta)]$ is a thesis of intuitionistic propositional calculus (see Porębska, Suchoń, 1996, s. 210).

[6] Hypothetical syllogism: $(\alpha \Rightarrow \beta) \Rightarrow [(\beta \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow \gamma)]$. This is a tautology of the intuitionistic propositional calculus (see Pogorzelski, 1975, p. 44).

[7] The formula $[\alpha \Rightarrow (\beta \Rightarrow \gamma)] \Rightarrow [(\alpha \wedge \beta) \Rightarrow \gamma]$ is a tautology of the intuitionistic propositional calculus (see Grzegorczyk, p. 106).

[8] The formula $\alpha \Rightarrow (\alpha \vee \beta)$ is a tautology of the intuitionistic propositional calculus (see Pogorzelski, p. 41).

[9] The formula $\beta \Rightarrow (\alpha \vee \beta)$ is a tautology of the intuitionistic propositional calculus (see Pogorzelski, p. 41).

9) $\{[F\alpha \Rightarrow F(\alpha \vee \beta)] \wedge [F\beta \Rightarrow F(\alpha \vee \beta)]\} \Rightarrow$
$[(F\alpha \vee F\beta) \Rightarrow F(\alpha \vee \beta)]$ – tautology[10]

10) $(F\alpha \vee F\beta) \Rightarrow F(\alpha \vee \beta)$ – 7, 8, 9, MP. □

The proof of the next fact is analogous.

T2': $(P\alpha \vee P\beta) \Rightarrow P(\alpha \vee \beta)$

Now we show, that axiom A6' is derivable from the axioms A2', A5' and some of tautologies of intuitionistic logic.

T3: $(H\alpha \wedge P\beta) \Rightarrow P(\alpha \wedge \beta)$

Proof

1) $\alpha \Rightarrow [\beta \Rightarrow (\alpha \wedge \beta)]$ – tautology[11]
2) $H\{\alpha \Rightarrow [\beta \Rightarrow (\alpha \wedge \beta)]\}$ – 1, RH
3) $H\{\alpha \Rightarrow [\beta \Rightarrow (\alpha \wedge \beta)]\} \Rightarrow$
$\{H\alpha \Rightarrow H[\beta \Rightarrow (\alpha \wedge \beta)]\}$ – A2'
4) $H\alpha \Rightarrow H[\beta \Rightarrow (\alpha \wedge \beta)]$ – 2, 3, MP
5) $H[\beta \Rightarrow (\alpha \wedge \beta)] \Rightarrow [P\beta \Rightarrow P(\alpha \wedge \beta)]$ – A5'
6) $H\alpha \Rightarrow [P\beta \Rightarrow P(\alpha \wedge \beta)]$ – 4, 5, hip.syll.[12], MP
7) $\{H\alpha \Rightarrow [P\beta \Rightarrow P(\alpha \wedge \beta)]\} \Rightarrow$
$[(H\alpha \wedge P\beta) \Rightarrow P(\alpha \wedge \beta)]$ – tautology[13]
7) $(H\alpha \wedge P\beta) \Rightarrow P(\alpha \wedge \beta)$ – 5, 6, MP

The axiom A6 is derivable from axioms A2, A5 and some of tautologies of intuitionistic logic.

T3': $(G\alpha \wedge F\beta) \Rightarrow F(\alpha \wedge \beta)$

Proof (Analogous to the proof of T1).

In this paper was shown that the set of axioms of IK_t can be reduced.

For the axiomatization of IK_t the following set of axioms is sufficent:

A1) All axioms of the intuitionistic propositional calculus

A2) $G(\alpha \Rightarrow \beta) \Rightarrow (G\alpha \Rightarrow G\beta)$ A2') $H(\alpha \Rightarrow \beta) \Rightarrow (H\alpha \Rightarrow H\beta)$

A3) $F(\alpha \vee \beta) \Leftrightarrow (F\alpha \vee F\beta)$ A3') $F(\alpha \vee \beta) \Leftrightarrow (F\alpha \vee F\beta)$

A4) $G(\alpha \Rightarrow \beta) \Rightarrow (F\alpha \Rightarrow F\beta)$ A4') $H(\alpha \Rightarrow \beta) \Rightarrow (P\alpha \Rightarrow P\beta)$

A5) $G\neg\alpha \Rightarrow \neg F\alpha$ A5') $H\neg\alpha \Rightarrow \neg P\alpha$

A6) $FH\alpha \Rightarrow \alpha$ A6') $PG\alpha \Rightarrow \alpha$

A7) $\alpha \Rightarrow GP\alpha$ A7') $\alpha \Rightarrow HF\alpha$

A8) $(F\alpha \Rightarrow G\beta) \Rightarrow G(\alpha \Rightarrow \beta)$ A8') $(P\alpha \Rightarrow H\beta) \Rightarrow H(\alpha \Rightarrow \beta)$

A9) $F(\alpha \Rightarrow \beta) \Rightarrow (G\alpha \Rightarrow F\beta)$ A9') $P(\alpha \Rightarrow \beta) \Rightarrow (H\alpha \Rightarrow P\beta)$

[10] The formula $[(\alpha \Rightarrow \gamma) \wedge (\beta \Rightarrow \gamma)] \Rightarrow [(\alpha \vee \beta) \Rightarrow \gamma]$ is a tautology of the intuitionistic propositional calculus (see Pogorzelski, p. 41).

[11] See note 5.

[12] See note 6.

[13] See note 7.

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Andrzej Malec

TWO FORMS OF LEGAL DEFINITION – A COMPARISON

I

In the paper I will emphasize some differences between two forms of definition which are used by the lawmaker in legal texts.

Definition 1.
The lawmaker is a fictitious person which is recognized as an author of all legal texts.

Definition 2.
A legal text is a text in which Statute Law is formulated.

The Civil Code of Poland is an example of a legal text.

Definition 3.
A legal definition is a part of a legal text in which the meaning of some legal terms is defined by the lawmaker.

In the methodology of law two kinds of legal definitions can be distinguished. We call them: „the intralanguage legal definitions" and „the metalanguage legal definitions".

The intralanguage legal definitions are expressed in the object language of a legal text.

Definition 4.
The object language of a legal text is a language in which the social environment is described and in which certain rules of behaviour are prescribed.

The intralanguage legal definitions describe legal constructions.

For example the definition:

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Paymaster is a person obliged to calculate and to gather taxes from tax-payers and to pay the gathered taxes to a tax--office

describes the legal construction of paymaster. I.e. this definition determines who should be recognized as a paymaster according to a certain system of law.

The definitions of the second kind, i.e. the metalanguage legal definitions are the definitions expressed in the metalanguage of a legal text.

Definition 5.

The metalanguage of a legal text is a language in which the object language of this text is described.

The metalanguage legal definitions are just expressions-replacement rules given by the lawmaker in the legal text.

For example the definition:

In the following text the word „goods" is used for denoting products or things purchased for sale

is the replacement rule for the expression „goods". I.e. according to this definition we are able to replace in a legal text all the appearances of the inscription „goods" by the inscription „products or things purchased for sale".

It is possible to identify the intralanguage definitions with the real definitions and to identify the metalanguage definitions with the nominal definitions. The support for such identification can be found both in legal and in logical works (for example in works of Bronislaw Wroblewski, Roman Suszko, Dimitr Gorski, Wolfgang Segeth). However, there are also other ideas of real and nominal definitions (for example these represented by Zygmunt Ziembinski, Tadeusz Kotarbinski, Kazimierz Ajdukiewicz, Leon Gumanski and others).

II

At the first sight the form of a legal definition does not matter.

For example, it is possible to replace the intralanguage definition of paymaster mentioned above:

Paymaster is a person obliged to calculate and to gather taxes from tax-payers and to pay the gathered taxes to a tax--office

by the metalanguage legal definition:

In the following text the word „paymaster" is used for denoting persons obliged to calculate and to gather taxes from tax-payers and to pay the gathered taxes to a tax-office.

The meaning of the term „paymaster" seems to be the same according to both definitions.

Similarly, it is possible to replace the metalanguage definition for the expression „goods" mentioned above:

In the following text the word „goods" is used for denoting products or things purchased for sale

by the intralanguage legal definition:

The goods are products or things purchased for sale.

Also in this case the meaning of the defined term (i.e. the term „goods") seems to be the same according to both definitions.

III

However, in some cases the form of definition may be essential for the result of interpretation of a legal text.

We can distinguish two differences between the intralanguage legal definitions and the metalanguage legal definitions which are important from the point of view of interpretation of a legal text.

First: the metalanguage legal definitions define the meaning of legal terms for the purpose of interpretation of that legal text in which they are placed only whereas the intralanguage legal definitions describe the constructions of a system of law so that – in principle – they are valid for legal terms employed in all the legal texts constituting this system of law.

For example: the intralanguage definition of paymaster mentioned above

Paymaster is a person obliged to calculate and to gather taxes from tax-payers and to pay the gathered taxes to a tax--office

determines who should be recognized as a paymaster according to a certain system of law – irrespective of any legal text.

On the contrary, the metalanguage legal definition for the expression „goods"

In the following text the word „goods" is used for denoting products or things purchased for sale

is explicitly restricted to the exactly one legal text. In other legal texts the expression „goods" may be defined in a different way.

The second important difference between the intralanguage legal definitions and the metalanguage legal definitions consists in the fact that the complex legal terms defined by metalanguage definitions are intensional whereas the complex terms defined by intralanguage definitions are extensional.

Definition 6.

A complex term is extensional if and only if the denotation of this term is completely determined by the denotations of its parts.

Definition 7.

A complex term is intensional if and only if the denotation of this term is not completely determined by the denotations of its parts.

Let us consider a complex legal term „the sale of goods".

Whenever we define what is the sale of goods by an intralanguage legal definition we thereby define what is the sale of products, since products are a kind of goods.

So, if we define the meaning of the legal term „the sale of goods" by an intralanguage legal definition, the defined term is extensional.

On the contrary, whenever we define how to replace the inscription „the sale of goods" we do not define how to replace the inscription „the sale of products". The inscription „the sale of goods" differs from the inscription „the sale of products", so, the replacement rules for the first inscription do not work for the second inscription.

So, if we define the meaning of the legal term „the sale of goods" by a metalanguage legal definition, the defined term is intensional.

The above differences between the intralanguage legal definitions and the metalanguage legal definitions affect the rules of interpretation of legal texts considered in the methodology of law. Thus, lawyers should recognize them.

However, the problem of intensionality of metalanguage legal definition is not sufficiently recognized in the methodology of law.

IV

There is also an important difference between the intralanguage definitions and the metalanguage definitions which is not recognized by most of the law theoreticians.

This difference concerns the defining of individual constants and occurs only in the process of formal reconstruction of a system of law.

In formal reconstruction of an intralanguage legal definition of an individual constant we must prove that the existence condition and the uniqueness condition are fulfilled.

For example:

Let a be an individual constant defined by a definition:

$$a = (\iota x)P(x)$$

where (ιx) is the iota-operator (we read: „x is the only object having the property P").

In this case we must prove that:

$(Ex)P(x)$	the existence condition,
$(x)(y)\{[P(x)\&P(y)] \Rightarrow (x = y)\}$	the uniqueness condition.

So, in the case of an intralanguage legal definition these conditions must be fulfilled.

On the contrary, it is not necessery to prove that the existence condition and the uniqueness condition are fulfilled when a metalanguage definition is considered. In fact, the reconstruction of a metalanguage legal definition for an individual constant must be treated as the defining of a predicate of the metalanguage of a legal text (The predicate in the form of: „... is denoted by the individual constant A").

Let us set forth an example.

In the process of formalization of the intralanguage definition of paymaster:

Paymaster is a person obliged to calculate and to gather taxes from tax-payers and to pay the gathered taxes to a tax--office

the term „paymaster" may be treated as a predicate denoting a property „to be a paymaster".

Interpretation:

Universum – the set of persons
$P(\ldots)$ – ... is a paymaster
$Q(\ldots)$ – ... is obliged to calculate taxes

$R(\ldots)$ – ... is obliged to gather taxes from tax-payers
$S(\ldots)$ – ... is obliged to pay the gathered taxes to a tax-office

Reconstruction of the definition:

$$(x)\{P(x) \Leftrightarrow [Q(x)\&R(x)\&S(x)]\}$$

In this case the existence and the uniqueness conditions do not apply.

However, in such case we are loosing the intuition that the paymaster is a certain legal construction: an abstract object in a legal system.

So, it is better to treat the term „paymaster" as an individual constant denoting a certain object in a legal system.

Interpretation:

Universum – the set of legal constructions
a – the legal construction of paymaster
$Q(\ldots)$ – ... contains the obligation to calculate taxes
$R(\ldots)$ – ... contains the obligation to gather taxes from tax-payers
$S(\ldots)$ – ... contains the obligation to pay the gathered taxes to a tax-office

Reconstruction of the definition:

$$a = (\iota x)[Q(x)\&R(x)\&S(x)]$$

where (ιx) is the iota-operator (we read: „x is the only object having together the properties Q, R and S").

But in this case the existence and the uniqueness conditions must be fulfilled.

On the other hand, when we formalize the metalanguage legal definition for the expression „paymaster":

In the following text the word „paymaster" is used for denoting persons obliged to calculate and to gather taxes from tax-payers and to pay the gathered taxes to a tax-office

we treat the term „paymaster" as a name of a certain expression of the object language of a legal text.

Interpretation:

Universum – the set of legal constructions
$P(\ldots)$ – ... is denoted by the individual constant A
$Q(\ldots)$ – ... contains the obligation to calculate taxes
$R(\ldots)$ – ... contains the obligation to gather taxes from tax-payers
$S(\ldots)$ – ... contains the obligation to pay the gathered taxes to a tax-office

Reconstruction of the definition:

$$(x)\{P(x) \Leftrightarrow [Q(x)\&R(x)\&S(x)]\}$$

Although we are speaking about legal constructions, the existence and the uniqueness conditions do not apply.

V

For the above reasons in the methodology of law the intralanguage legal definitions and the metalanguage legal definitions should be distinguished.

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